

Optimal Taxes on Capital in the OLG Model with Uninsurable Idiosyncratic Income Risk*

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Abstract

We characterize the optimal linear tax on capital in an Overlapping Generations model with two period lived households facing uninsurable idiosyncratic labor income risk. The Ramsey government internalizes the general equilibrium effects of private precautionary saving on factor prices. For logarithmic utility a complete analytical solution of the Ramsey problem exhibits an optimal aggregate saving rate that is independent of income risk, whereas the optimal time-invariant tax on capital implementing this saving rate is increasing in income risk. The optimal saving rate is constant along the transition and its sign depends on the magnitude of risk and on the Pareto weight of future generations. If the Ramsey tax rate that maximizes steady state utility is positive, then implementing this tax rate permanently induces a Pareto-improving transition even if the initial equilibrium is dynamically efficient. For general Epstein-Zin-Weil utility we show that the optimal steady state saving rate is increasing in income risk if and only if the intertemporal elasticity of substitution is smaller than 1.

Keywords: Idiosyncratic Risk, Taxation of Capital, Overlapping Generations, Precautionary Saving, Pecuniary Externality

J.E.L. classification codes: H21, H31, E21

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1 Introduction

How should a benevolent government tax capital in a production economy when households face uninsurable idiosyncratic labor income risk and engage in precautionary saving against this risk? Partial answers to this question have been given in Bewley style general equilibrium models with neoclassical production and infinitely lived consumers, starting from Aiyagari (1995)’s characterization of the optimal steady state capital income tax rate, and continuing with recent work providing characterizations of the optimal path of capital income taxes by Panousi and Reis (2015, 2017), Gottardi et al. (2015), Dyrda and Pedroni (2018), Açıkgöz et al. (2018), Chen et al. (2019) and Chien and Wen (2019).

In this paper we complement and extend this literature by providing an analytical characterization of optimal linear taxes on capital in a canonical Diamond (1965) style Overlapping Generations model with uninsurable idiosyncratic labor income risk in the second period of life. The Ramsey government (Ramsey 1927) has to respect equilibrium behavior of private agents, uses the tax revenues from the tax on capital to finance lump-sum transfers to households and maximizes a social welfare function with arbitrary Pareto weights on different generations born into this economy.

For logarithmic utility we provide a complete analytical solution of the optimal dynamic Ramsey allocation and associated tax policy along the transition. It is characterized by a time-invariant aggregate saving rate s , defined as the share of aggregate labor income devoted to capital accumulation. This saving rate is independent of the magnitude of idiosyncratic income risk, and can be implemented as a competitive equilibrium with a proportional tax on capital that is also constant over time and strictly increasing in the extent of income risk. As key benefit of the analytical solution we show explicitly that the optimal saving rate chosen by the Ramsey government is shaped by three distinct effects. First, an increase in the saving rate reduces consumption when young and increases it when old, holding factor prices constant. We term this the *partial equilibrium* $PE(s)$ effect, which is fully taken into account by private households when making savings decisions and taking factor prices as given. Second, a larger saving rate of the young today raise the capital stock and thus increase wages and lower returns in general equilibrium, an effect we call the *current generations* $CG(s)$ effect. Third, a higher saving rate today has a general equilibrium *future generations* $FG(s)$ effect, since a higher current saving rate increases the future capital stock, future wages in general equilibrium and thus impacts welfare of future generations. With log-utility we derive all effects in closed form to show that income risk

does not affect the optimal saving rate chosen by the Ramsey planner, because the effect of risk on the general equilibrium effect exactly offsets the impact of risk on the partial equilibrium precautionary savings effect for each generation.

To interpret this finding, consider an increase in income risk. Households respond by increasing their saving rate due to the precautionary savings motive. But this increase in the saving rate raises wages and thus the risky income component in the next period, while lowering capital returns and the associated income. Since labor income risk is uninsurable by assumption, this additional wage risk is welfare reducing. In contrast, the Ramsey planner internalizes this negative side effect from private precautionary saving when setting tax rates on capital, and with log-utility, exactly offsets the $PE(s)$ effect through the $CG(s)$ effect.¹ The benevolent Ramsey government implements the optimal allocation by offsetting the negative precautionary savings externality through taxes on capital, thereby reducing the saving rate and capital formation. Hence, more broadly, our optimal tax result is shaped by the Pigouvian taxation principle (Pigou 1920) aimed at correcting externalities. Since the individually chosen (socially suboptimal) saving rate is increasing in income risk, so is the tax rate on capital correcting the externality from these choices.

If the Ramsey government additionally values future generations, as in the current generations effect, the future generations effect internalizes the general equilibrium feedback on returns, wages and thus exposure to idiosyncratic wage risk for future generations from a change in the current saving rate. With logarithmic utility all these future risk terms again cancel out, and the saving rate is not affected by income risk at all. Thus, future generations unambiguously benefit from a higher capital stock which pushes up the optimal saving rate desired by the Ramsey planner, relative to the world with only one generation. The presence of the future generations effect implies that the tax rate implementing the optimal allocation may therefore be positive or negative, depending on how strongly the Ramsey government values future relative to current generations.

Our canonical OLG model permits us to connect the results on optimal taxation of capital to the classical discussion of dynamic efficiency, and we establish a somewhat surprising result. Consider the optimal Ramsey tax rate when the government places all weight in the social welfare function on generations living in the steady state and the future generations effect is maximally potent. If this tax rate is positive (which is true if income risk is

¹Some papers emphasize that private precautionary savings behavior induced by idiosyncratic income risk creates a pecuniary externality with first order welfare implications in incomplete markets models, see e.g. Davila et al. (2012) or Park (2018). In this paper we refer to this effect on current as well as on future factor prices as *general equilibrium effect*. Occasionally, we also label it a *precautionary savings externality*.

sufficiently high), then a government implementing this constant tax rate along the transition generates a Pareto-improving transition from the unregulated steady state equilibrium. This holds true even if the original equilibrium is dynamically efficient and thus the tax on capital reduces aggregate consumption along the transition path. The steady state utility maximizing tax rate takes into account the welfare losses induced by the crowding out of capital. Since the capital stock monotonically decreases along the transition, welfare losses from this crowding out effect monotonically increase along the transition. Instead, the utility gains from a tax-induced reduction of the saving rate (and thus higher first period consumption) are highest in the initial period, decrease monotonically along the transition, and still dominate the welfare losses from crowding out in the long-run. Consequently, setting the tax rate in all periods to the long-run welfare maximizing rate induces welfare gains for all transitional generations and thus constitutes a Pareto improvement.

In the last part of the paper we extend the steady state results to Epstein-Zin-Weil utility (EZW utility, see Epstein and Zin (1989, 1991) and Weil (1989)). We first show that our closed form results for the transition go through unchanged for arbitrary risk aversion if the inter-temporal elasticity of substitution (IES) is equal to one. Next, we demonstrate that the optimal steady state saving rate is increasing in the amount of income risk if and only if the IES is smaller than 1. With EZW utility the objective of households (and thus the Ramsey government) is to maximize utility from safe consumption when young and from the certainty equivalent of utility from risky consumption when old. When risk increases, the certainty equivalent from consumption when old decreases. In response the government finds it optimal to increase mean old age consumption by increasing the saving rate if the willingness to inter-temporally substitute consumption is relatively low, with log-utility ($IES = 1$) serving as the watershed case. The associated optimal steady state tax rate implementing this saving rate is increasing in income risk unless both the IES and risk aversion (RA) are large, in which case the Ramsey tax rate might be *declining* in income risk. A necessary condition for this result is that households in the laissez faire equilibrium decrease their private saving rate in response to increased income risk. They may choose to do so if they have high RA and high IES because of the low utility value from old-age consumption (high RA) and the high willingness to inter-temporally substitute consumption (high IES) in response to an increase of risk. The Ramsey government internalizes the associated feedback on capital formation through the *future generations effect* and may therefore find it optimal to dampen the private household saving reaction by cutting the tax on capital in response to an increase in income risk.

This paper contributes to the literature studying optimal allocations and optimal Ramsey taxation in models with uninsurable idiosyncratic income risk. The first strand of this literature analyzes the role of uninsurable idiosyncratic labor income risk for capital accumulation and optimal capital income taxation in infinite horizon Aiyagari (1994), Bewley (1986), İmrohoroglu (1989) and Huggett (1993) economies. Davila et al. (2012) is most related to our work. They characterize constrained efficient allocations where a planner directly chooses allocations, but cannot transfer resources between households with different shock realizations to provide direct insurance.² The paper emphasizes three determinants of the optimal allocation: how uninsurable risk affects private precautionary savings, how general equilibrium prices affect the total income risk of a consumer as well as how the distribution of incomes, in particular the income composition of consumption- and wealth-poor agents in the economy, affect aggregate welfare.³ We in contrast study an OLG economy where the distribution of factor incomes across generations, rather than at a given point of time, is crucial for the determination of optimal policy. To obtain closed form solutions we abstract from within-generation heterogeneity so that inter-generational distribution is the only distributional effect in the model. In addition, we characterize the optimal solution of the Ramsey tax problem with linear taxes on capital, rather than focusing on constrained efficient allocations. However, we show that with our choice of policy instruments, the Ramsey government can in fact implement constrained efficient allocations.

The work on optimal Ramsey capital taxation in Bewley models starts with Aiyagari (1995). Under the assumptions that government spending is endogenous and that the optimal allocation converges to a stationary equilibrium, he argues that in this stationary equilibrium the capital income tax is positive and restores the modified golden rule.⁴ Recent work by Chen et al. (2019) reassesses Aiyagari (1995)'s main finding of positive capital income taxes in models with exogenous government spending, as in the standard Ramsey optimal taxation literature. Depending on the IES there either is no Ramsey steady state

²The notion of constrained efficiency follows Diamond (1967) and Geanakoplos and Polemarchakis (1986), and refers to a planner problem with the constraint that the planner cannot directly overcome a friction implied by missing markets.

³In Davila et al. (2012) asset-income poor households benefit from an increase of the capital stock and thus wages. Park (2018) introduces endogenous human capital accumulation so that welfare of human-capital poor households might be improved by lower wages, which adds an additional distribution effect, with welfare implications of changing factor incomes opposite to those by Davila et al. (2012).

⁴Chamley (2001) develops a partial equilibrium model to clarify that the Chamley-Judd (Judd 1985; Chamley 1986) result of zero optimal capital taxes depends on the assumption of complete markets and breaks down if households face income risk and a borrowing constraint. In Chamley (2001)'s partial equilibrium analysis, the general equilibrium effects that are crucial to our results are missing by construction.

with interest rate lower than the discount rate, or the Lagrange multiplier on the resource constraint diverges in that steady state. In both cases Aiyagari (1995)’s argument establishing an optimal long-run positive capital income tax does not extend to the canonical infinite horizon incomplete markets model with exogenous government spending.^{5,6} In our OLG model we can characterize, for an IES of one, the entire time path of optimal Ramsey allocations analytically, and thus can demonstrate that the allocation indeed converges to a steady state. Furthermore we obtain a complete characterization of optimal capital tax rates along the transition.⁷ A related theoretical literature studies optimal capital income taxes in models with idiosyncratic *investment* risk, see Evans (2015), Panousi (2015), and Panousi and Reis (2017). Their key focus is on the role of capital income taxes in providing insurance or redistribution; none of these papers emphasizes the role of general equilibrium feedback from precautionary saving behavior on optimal capital income taxation.

Finally, our work connects to the literature on optimal capital income taxation in life-cycle economies, as in the two-period models of Pestieau (1974) and Atkinson and Sandmo (1980). Erosa and Gervais (2001, 2002), Conesa et al. (2009), Garriga (2017) and Peterman (2016) extend these studies to multiple periods and emphasize that capital income taxes are only zero under strong assumptions on preferences, or if labor income tax rates can depend on household age. The general equilibrium price effects of precautionary savings on prices in the Ramsey problem are not addressed in these papers.

Section 2 presents the model and Section 3 characterizes the competitive equilibrium. Section 4 lays out the Ramsey problem and presents the analytical solution for log-utility. Section 5 discusses the efficiency properties of the Ramsey equilibrium and gives conditions under which implementing the long-run optimal policy induces a Pareto improving transition. Section 6 generalizes the results to EZW utility and Section 7 concludes.

⁵In related work, Chien and Wen (2019) develop a tractable Aiyagari-Bewley-Huggett model with preference rather than productivity shocks to address the impact of precautionary saving, through the general equilibrium interest rate, on the fraction of households at the borrowing constraint.

⁶Heathcote, Storesletten, and Violante (2017) also develop an analytically tractable model with idiosyncratic income risk. They focus on characterizing the optimal progressivity of labor income taxation in a model with infinitely lived households, endogenous labor supply but without capital.

⁷Quantitative work in infinite horizon economies by Dyrda and Pedroni (2018) and Açıkgöz et al. (2018) analyze optimal fiscal policy along the economy’s transition from the status quo to the long-run steady state and find robustly positive capital income taxes. A similar finding is obtained by Gottardi et al. (2015) in a model with risky human capital originally proposed by Krebs (2003). These papers extend the work by Domeij and Heathcote (2004) analyzing the welfare consequences of abolishing capital income taxes in a Aiyagari-Bewley-Huggett economy taking into account the transition. Whereas idiosyncratic labor income risk plays a key role in these papers, none of them emphasizes how the general equilibrium price effects affect the optimal allocation chosen by the Ramsey planner as we do.

2 Model

Time is discrete and extends from $t = 0$ to $t = \infty$. In each period a new generation is born that lives for two periods. Thus at any point in time there is a *young* and an *old* generation. We normalize household size to 1 for each age cohort. In addition there is an initial old generation that has one remaining year of life.

2.1 Household Preferences and Endowments

2.1.1 Endowments

Each household has one unit of time in both periods, supplied inelastically to the market. Labor productivity when young is equal to $(1-\kappa)$, and, as in Harenberg and Ludwig (2015), in the second period labor productivity is given by $\kappa\eta_{t+1}$, where $\kappa \in [0, 1)$ is a parameter that captures relative labor income of the old, and η_{t+1} is an idiosyncratic labor productivity shock. We assume that the cdf of η_{t+1} is given by $\Psi(\eta_{t+1})$ in every period and denote the corresponding pdf by $\psi(\eta_{t+1})$. We assume that Ψ is both the population distribution of η_{t+1} as well as the cdf of the productivity shock for any given individual (that is, we assume a Law of Large Numbers, LLN henceforth). Whenever there is no scope for confusion we suppress the time subscript of the productivity shock η_{t+1} . We make the following

Assumption 1. *The shock η_{t+1} takes positive values Ψ -almost surely and*

$$\int \eta_{t+1} d\Psi = 1.$$

Each member of the initial old generation is additionally endowed with assets equal to a_0 , equal to the initial capital stock k_0 in the economy. The asset endowment is independent of the household's realization of the shock η .

2.1.2 Preferences

A household of generation $t \geq 0$ has preferences over consumption allocations $c_t^y, c_{t+1}^o(\eta_{t+1})$ given by

$$V_t = u(c_t^y) + \beta \int u(c_{t+1}^o(\eta_{t+1})) d\Psi. \quad (1)$$

Lifetime utility of the initial old generation is determined as

$$V_{-1} = \int u(c_0^o(\eta_0)) d\Psi.$$

In order to obtain the sharpest analytical results in the first part of the paper we will assume logarithmic utility:

Assumption 2. *The utility function u is logarithmic, $u(c) = \ln(c)$.*

We will generalize our results to a Epstein-Zin-Weil (Epstein and Zin 1989; Epstein and Zin 1991; Weil 1989) utility function, which nests constant relative risk aversion (CRRA) preferences, in Section 6 of the paper.

2.2 Technology

The representative firm operates the Cobb-Douglas production technology:

$$F(K_t, L_t) = K_t^\alpha (L_t)^{1-\alpha}.$$

Furthermore we assume that capital fully depreciates between two (30 year) periods.

2.3 Government

The government levies a potentially time varying tax τ_t on capital, and rebates the proceeds in a lump-sum fashion to all members of the current old generation as a transfer T_t . Note that the restriction that transfers accrue exclusively to old households implies that the government has no direct tool for intergenerational redistribution.⁸ We assume that the government has the following social welfare function

$$W = \sum_{t=-1}^{\infty} \omega_t V_t,$$

⁸It also implies that, conditional on a beginning of the period capital stock given by past household decisions, the government cannot alter lifetime utility of the newborn generation in period t through changing the current tax τ_t . And since tax revenues from the current old are fully rebated back to this generation, remaining lifetime utility of the old is unaffected by the tax τ_t . This in turn insures that the government has no incentive to deviate, in period t , from the period zero tax plan $\{\tau_t\}$. In other words, given the restriction on the set of policies, Ramsey tax policies will be time-consistent in our environment.

where $\{\omega_t\}_{t=-1}^{\infty}$ are the Pareto weights on different generations and satisfy $\omega_t \geq 0$. Since lifetime utilities of each generation will be bounded, so will be the social welfare function as long as $\sum_{t=-1}^{\infty} \omega_t < \infty$. We will also consider the case $\omega_t = 1$ for all t , in which case we will take the social welfare function to be defined as

$$W = \lim_{T \rightarrow \infty} \frac{\sum_{t=-1}^T V_t}{T},$$

which is equivalent to maximizing steady state welfare.

2.4 Competitive Equilibrium

2.4.1 Household Budget Set and Optimization Problem

The budget constraints in both periods read as

$$c_t^y + a_{t+1} = (1 - \kappa)w_t \tag{2a}$$

$$c_{t+1}^o = a_{t+1}R_{t+1}(1 - \tau_{t+1}) + \kappa\eta_{t+1}w_{t+1} + T_{t+1}, \tag{2b}$$

where w_t, w_{t+1} are the aggregate wages in period t and $t + 1$, $R_{t+1} = 1 + r_{t+1}$ is the gross interest rate between period t and $t + 1$, and T_{t+1} are lump-sum transfers to the old generation, and η_{t+1} is the age-2 period- $t + 1$ idiosyncratic shock to wages.⁹

2.4.2 Firm Optimization

From the firm's first order conditions we get

$$R_t = \alpha k_t^{\alpha-1} \tag{3a}$$

$$w_t = (1 - \alpha)k_t^{\alpha} \tag{3b}$$

where

$$k_t = \frac{K_t}{L_t} = \frac{K_t}{1 - \kappa + \kappa \int \eta_t d\Psi} = K_t$$

is the capital-labor ratio. Since $L_t = 1$, we henceforth do not need to distinguish between the aggregate capital stock K_t and the capital-labor ratio.

⁹Notice that instead of working with a tax on capital τ_t , one could work, completely equivalently, with standard capital income taxes τ_t^k . We discuss this equivalence in detail in Section 4.4 of the paper.

2.4.3 Equilibrium Definition

We are now ready to define a competitive equilibrium.¹⁰

Definition 1. *Given the initial condition $a_0 = k_0$ an allocation is a sequence $\{c_t^y, c_t^o(\eta_t), L_t, a_{t+1}, k_{t+1}\}_{t=0}^\infty$.*

Definition 2. *Given the initial condition $a_0 = k_0$ and a sequence of tax policies $\tau = \{\tau_t\}_{t=0}^\infty$, a competitive equilibrium is an allocation $\{c_t^y, c_t^o, L_t, a_{t+1}, k_{t+1}\}_{t=0}^\infty$, prices $\{R_t, w_t\}_{t=0}^\infty$ and transfers $\{T_t\}_{t=0}^\infty$ such that*

1. *given prices $\{R_t, w_t\}_{t=0}^\infty$ and policies $\{\tau_t, T_t\}_{t=0}^\infty$ for each $t \geq 0$, $(c_t^y, c_{t+1}^o(\eta_{t+1}), a_{t+1})$ maximizes (1) subject to (2a) and (2b) (for each realization of η_{t+1});*
2. *consumption $c_0^o(\eta_0)$ of the initial old satisfies (2b) (for each realization of η_0):*

$$c_0^o = a_0 R_0 (1 - \tau_0) + \kappa \eta_0 w_0 + T_0;$$

3. *prices satisfy equations (3a) and (3b);*
4. *the government budget constraint is satisfied in every period: for all $t \geq 0$*

$$T_t = \tau_t R_t k_t;$$

5. *markets clear*

$$\begin{aligned} L_t &= L = 1 \\ a_{t+1} &= k_{t+1} \\ c_t^y + \int c_t^o(\eta_t) d\Psi + k_{t+1} &= k_t^\alpha. \end{aligned}$$

Denote by $W(\tau)$ social welfare associated with an equilibrium for given tax policy τ . As we will show below, for a given tax policy τ the associated competitive equilibrium in our economy exists and is unique and thus the function $W(\tau)$ is well-defined as long as $\tau_t \in (-\infty, 1)$ for all t .

¹⁰Since our main results below will focus on economies that are dynamically efficient, we have thus far implicitly assumed that the only asset households can trade is physical capital, thereby ruling out equilibria with bubbles initiated by the initial old generation, or by the government issuing fiat money. Our definition of equilibrium reflects this focus. For the same reason we also abstract from a pay-as-you-go social security system as part of the fiscal instruments at the disposal of the government.

Definition 3. Given the initial condition $a_0 = k_0$, a Ramsey equilibrium is a sequence of tax policies $\hat{\tau} = \{\hat{\tau}_t\}_{t=0}^{\infty}$ and equilibrium allocations, prices and transfers associated with $\hat{\tau}$ (in the sense of the previous definition) such that

$$\hat{\tau} \in \arg \max_{\tau} W(\tau).$$

3 Analysis of Equilibrium for a Given Tax Policy

3.1 Partial Equilibrium

We first analyze the household problem for given prices and policies. We proceed under the assumption that a unique solution characterized by the Euler equation exists, and then make sufficient parametric assumptions to insure that this is indeed the case.

The optimal asset choice a_{t+1} satisfies

$$1 = \beta(1 - \tau_{t+1}) \int \frac{R_{t+1} [u'(a_{t+1}R_{t+1}(1 - \tau_{t+1}) + \kappa\eta_{t+1}w_{t+1} + T_{t+1})]}{u'((1 - \kappa)w_t - a_{t+1})} d\Psi(\eta_{t+1}).$$

Defining the saving rate as

$$s_t = \frac{a_{t+1}}{(1 - \kappa)w_t}$$

we can rewrite the above equation as

$$1 = \beta(1 - \tau_{t+1}) \int \frac{R_{t+1} [u'(s_t R_{t+1}(1 - \tau_{t+1})(1 - \kappa)w_t + \kappa\eta_{t+1}w_{t+1} + T_{t+1})]}{u'[(1 - \kappa)w_t(1 - s_t)]} d\Psi(\eta_{t+1}), \quad (4)$$

which defines the solution

$$s_t = s_t(w_t, w_{t+1}, R_{t+1}, \tau_{t+1}, T_{t+1}; \beta, \kappa, \Psi).$$

Note by assumption 1 that consumption in the second period is positive Ψ -almost surely. Without further assumptions on the fundamentals we cannot make analytical progress. Therefore now invoke assumption 2 that the utility function is logarithmic. Then the Euler equation becomes:

$$1 = \beta(1 - \tau_{t+1}) \int \frac{1 - s_t}{s_t(1 - \tau_{t+1}) + \frac{\kappa w_{t+1}}{(1 - \kappa)w_t R_{t+1}} \eta_{t+1} + \frac{T_{t+1}}{(1 - \kappa)w_t R_{t+1}}} d\Psi(\eta_{t+1}). \quad (5)$$

Equation (5) implicitly defines the optimal partial equilibrium saving rate $s_t = s(w_t, w_{t+1}, R_{t+1}, \tau_{t+1}, T_{t+1}; \beta, \kappa \Psi)$.

3.2 General Equilibrium

Now we exploit the remaining equilibrium conditions. In equilibrium factor prices and transfers are given by

$$w_t = (1 - \alpha)k_t^\alpha \quad (6a)$$

$$R_{t+1} = \alpha k_{t+1}^{\alpha-1} \quad (6b)$$

$$T_{t+1} = \tau_{t+1} R_{t+1} k_{t+1} \quad (6c)$$

From the definition of the saving rate $s_t = \frac{a_{t+1}}{(1-\kappa)w_t}$, equation (6a) and market clearing in the asset market, which implies $a_{t+1} = k_{t+1}$, we find that

$$k_{t+1} = s_t(1 - \kappa)(1 - \alpha)k_t^\alpha \quad (7)$$

In general, for a given sequence of capital taxes $\{\tau_t\}_{t=0}^\infty$ the competitive equilibrium is a sequence of capital stocks $\{k_{t+1}\}_{t=0}^\infty$ that solves, for a given initial condition k_0 , the first order difference equation (5) when factor prices have been substituted

$$1 = \alpha\beta(1 - \tau_{t+1}) \left(\frac{(1 - \kappa)(1 - \alpha)k_t^\alpha - k_{t+1}}{k_{t+1}} \right) \Gamma, \quad (8)$$

where the constant

$$\Gamma = \int (\kappa\eta_{t+1}(1 - \alpha) + \alpha)^{-1} d\Psi(\eta_{t+1}) = \Gamma(\alpha, \kappa; \Psi) \quad (9)$$

fully captures the impact of idiosyncratic income risk on the equilibrium dynamics of the capital stock.

Equation (8) implicitly defines the function $k_{t+1} = \Omega(k_t, \tau_{t+1})$. Alternatively, and often more conveniently, instead of expressing the solution as $k_{t+1} = \Omega(k_t, \tau_{t+1})$, we can also express it in terms of the saving rate as

$$s_t = \frac{k_{t+1}}{(1 - \alpha)(1 - \kappa)k_t^\alpha} = \frac{\Omega(k_t, \tau_{t+1})}{(1 - \alpha)(1 - \kappa)k_t^\alpha} = \Lambda(k_t, \tau_{t+1}) \quad (10)$$

where the function $s_t = \Lambda(k_t, \tau_{t+1})$ solves (using the definition of the saving rate in equation (8)):

$$1 = \alpha\beta(1 - \tau_{t+1}) \left(\frac{1 - s_t}{s_t} \right) \Gamma. \quad (11)$$

3.3 Characterization of the Saving Rate

Evidently, equation (11) has a closed form solution for the saving rate s_t in general equilibrium, and we can give a complete analytical characterization of its comparative statics properties.

Proposition 1. *Suppose assumptions 1 and 2 are satisfied. Then for all $k_t > 0$ and all $\tau_{t+1} \in (-\infty, 1]$ the unique saving rate $s_t = \Lambda(k_t, \tau_{t+1}; \Gamma)$ is given by*

$$s_t = \frac{1}{1 + [(1 - \tau_{t+1})\alpha\beta\Gamma(\alpha, \kappa; \Psi)]^{-1}}, \quad (12)$$

which is strictly increasing in Γ , strictly decreasing in τ_{t+1} and independent of the beginning of the period capital stock.

The next corollary assures that any desired saving rate $s_t \in (0, 1]$ can be implemented as part of a competitive equilibrium by appropriate choice of the capital tax rate τ_{t+1} . This corollary is crucial for our approach of solving the optimal Ramsey tax problem, since we can cast that problem directly in terms of the government choosing saving rates rather than tax rates.

Corollary 1. *For each saving rate $s_t \in (0, 1]$ there exists a unique tax rate $\tau_{t+1} \in (-\infty, 1)$ that implements that saving rate s_t as part of a competitive equilibrium.*

Finally we want to determine the influence of income risk on the saving rate in general equilibrium. From Proposition 1 we know that the saving rate depends on income risk η exclusively through the constant Γ . Furthermore, Γ is a strictly convex function of income risk η , and thus by Jensen's inequality we have the following:

Observation 1. *Assume that $\alpha \in (0, 1)$ and $\kappa > 0$. Then*

1. *The constant $\Gamma(\alpha, \kappa; \Psi)$ is strictly increasing in the amount of income risk, in the sense that if the distribution $\tilde{\Psi}$ over η is a mean-preserving spread of Ψ , then $\Gamma(\alpha, \kappa; \Psi) < \Gamma(\alpha, \kappa; \tilde{\Psi})$.*

2. Define the degenerate distribution at $\eta \equiv 1$ as $\bar{\Psi}$, then for any nondegenerate Ψ

$$1 < \bar{\Gamma} := \Gamma(\alpha, \kappa; \bar{\Psi}) < \Gamma(\alpha, \kappa; \Psi)$$

We can immediately deduce the following:

Corollary 2. *The equilibrium saving rate is strictly increasing in the amount of income risk.*

The proof of this result follows directly from the fact that $s_t = \Lambda(k_t, \tau_{t+1}; \Gamma)$ is strictly increasing in Γ and Γ is strictly increasing in the amount of income risk. Equipped with this full characterization of the competitive equilibrium for a given sequence of tax policies $\{\tau_{t+1}\}_{t=0}^{\infty}$ we now turn to the analysis of *optimal fiscal policy*.

4 The Ramsey Problem

The objective of the government is to maximize social welfare $W(k_0) = \sum_{t=-1}^{\infty} \omega_t V_t$ by choice of capital taxes $\{\tau_{t+1}\}_{t=0}^{\infty}$ where V_t is the lifetime utility of generation t in the competitive equilibrium associated with the sequence $\{\tau_{t+1}\}_{t=0}^{\infty}$. We start with general expected utility preferences and later again invoke assumption 2 that the utility function is logarithmic. Making use of Corollary 1 we can substitute out taxes to write lifetime utility in terms of the saving rate s_t yielding

$$V(k_t, s_t) = u((1 - s_t)(1 - \kappa)(1 - \alpha)k_t^\alpha) + \beta \int u(\kappa \eta_{t+1} w(s_t) + R(s_t) s_t (1 - \kappa)(1 - \alpha)k_t^\alpha) d\Psi(\eta_{t+1}), \quad (13)$$

where

$$w(s_t) = (1 - \alpha) [k_{t+1}(s_t)]^\alpha \quad (14a)$$

$$R(s_t) = \alpha [k_{t+1}(s_t)]^{\alpha-1} \quad (14b)$$

$$k_{t+1}(s_t) = s_t (1 - \kappa)(1 - \alpha)k_t^\alpha. \quad (14c)$$

We could now substitute factor prices in the lifetime utility function, but for the purpose of better interpretation of the results we refrain from doing so at this point.

Finally, remaining lifetime utility of the initial old generation is given by (with factor prices already substituted out)

$$V_{-1} = V(k_0, \tau_0) = \int u([\alpha + \kappa\eta_0(1 - \alpha)] k_0^\alpha) d\Psi(\eta_0) = V(k_0) \quad (15)$$

Note that τ_0 is irrelevant for welfare of the initial old generation (and all future generations). This is due to the fact that, since k_0 is a fixed initial condition, τ_0 is nondistortionary, is lump-sum rebated and that the government is assumed to have a period-by-period budget balance. In fact, expression (15) shows that with the set of policies we consider lifetime utility of the initial old cannot be affected at all, which is useful since we therefore do not need to include it in the social welfare function.¹¹

By Corollary 1 the Ramsey government can implement any sequence of saving rates $\{s_t\}_{t=0}^\infty$ as a competitive equilibrium and thus can choose private saving rates directly. We can therefore restate the problem the Ramsey government solves for $\sum_{t=0}^\infty \omega_t < \infty$ as¹²

$$W(k_0) = \max_{\{s_t\}_{t=0}^\infty} \sum_{t=0}^\infty \omega_t V(k_t, s_t) \quad (16)$$

subject to (14a)–(14c).

In the remainder of this section we fully characterize the solution to the Ramsey problem. We can do so for arbitrary social welfare weights $\{\omega_t\}_{t=0}^\infty$ using the sequential formulation of the problem, as Appendix B shows. In the main text we exploit the recursive formulation of the problem, which requires a stationarity assumption on the social welfare weights (Assumption 3 below), but allows us to derive and interpret the solution in the most transparent manner.

4.1 Recursive Formulation and Characterization of Ramsey Problem

The Ramsey problem lends itself to a recursive formulation, under the following assumption on the social welfare weights:

¹¹For a given capital stock k_t , the same argument applies to an arbitrary old generation at period t , in that remaining lifetime utility of this old generation cannot any longer be affected by τ_t . Since the same is true for lifetime utility of newborns in period t , the government has no incentives to ex post (after capital k_t is installed) deviate from its period zero Ramsey plan, in contrast to the typical time consistency problem often encountered in the optimal capital income tax literature. This fact also implies that we can write the Ramsey problem recursively, as done in the next subsection.

¹²Recall that for $\omega_t = 1$ in all t we accordingly have $W(k_0) = \max_{\{s_t\}_{t=0}^\infty} \lim_{T \rightarrow \infty} \frac{\sum_{t=0}^T V(k_t, s_t)}{T}$.

Assumption 3. *The social welfare weights satisfy, for all $t \geq 0$, $\omega_t > 0$ and*

$$\frac{\omega_{t+1}}{\omega_t} = \theta \in (0, 1).$$

Under this assumption, the recursive formulation of the problem reads as

$$\begin{aligned} W(k) = & \max_{s \in [0,1]} u((1-s)(1-\kappa)(1-\alpha)k^\alpha) \\ & + \beta \int u(\kappa\eta w(s) + R(s)s(1-\kappa)(1-\alpha)k^\alpha) d\Psi(\eta) + \theta W(k'(s)) \end{aligned} \quad (17)$$

s.t.

$$k'(s) = s(1-\kappa)(1-\alpha)k^\alpha \quad (18a)$$

$$R(s) = \alpha [k'(s)]^{\alpha-1} \quad (18b)$$

$$w(s) = (1-\alpha) [k'(s)]^\alpha \quad (18c)$$

This way of writing the problem highlights the three effects the Ramsey government considers when choosing the current saving rate s and thus the tax rate on capital τ that then induces private households to choose the saving rate s . First, holding equilibrium prices constant, there is the direct effect of reduced consumption when young and increased consumption when old, henceforth denoted by $PE(s)$. Private households, when making their decisions, fully take this effect into account. Second, a change in the saving rate s impacts the current generation through general equilibrium effects of changed wages when young and interest rates when old. We denote this effect as $CG(s)$. Third, a change in the current saving rate increases the future capital stock and thus impacts lifetime utility of future generations, an effect we denote as $FG(s)$.

Taking first order conditions for problem (17) yields:

$$\begin{aligned} 0 = & (1-\kappa)(1-\alpha)k^\alpha \left[-u'(c^y) + R(s)\beta \int u'(c^o(\eta)) d\Psi(\eta) \right] \\ & + \beta \int u'(c^o(\eta)) [\kappa\eta w'(s) + (1-\kappa)(1-\alpha)k^\alpha R'(s)s] d\Psi(\eta) \\ & + \theta W'(k'(s)) \frac{dk'(s)}{ds} \\ = & PE(s) + CG(s) + FG(s) \end{aligned}$$

We make the following observations:

1. Denoting by s^{CE} the saving rate households would choose in the competitive equilibrium with zero capital taxes, we have $PE(s^{CE}) = 0$.
2. Appendix A demonstrates that the current generations general equilibrium equals

$$CG(s) = (1 - \alpha)\alpha [(1 - \kappa)(1 - \alpha)k^\alpha]^\alpha [s]^{\alpha-1} \beta \int u'(c^o(\eta)) [\kappa\eta - 1] d\Psi(\eta).$$

The magnitude of a change in factor prices induced by a change in saving rates is purely determined from the production side of the economy (the first term in the product). The utility value to the household and thus to the Ramsey government of these factor price movements, however, depends on the utility function since it determines the size of the covariance between $u'(c^o(\eta))$ and η . We note that

$$\begin{aligned} \int u'(c^o(\eta)) [\kappa\eta - 1] d\Psi(\eta) &= (\kappa - 1) \int u'(c^o(\eta)) d\Psi(\eta) + Cov[u'(c^o(\eta)), (\kappa\eta - 1)] \\ &< (\kappa - 1) \int u'(c^o(\eta)) d\Psi(\eta) < 0 \end{aligned}$$

and thus the current generation general equilibrium effect is negative, driving down the desired saving rate for the Ramsey government. Higher wages exacerbate idiosyncratic income and thus consumption risk and therefore it is optimal for the social planner to reduce labor income risk by reducing savings incentives, other things equal.

The fact that the current generations effect is negative has immediate consequences for optimal tax policy in the case that $\theta = 0$ so that the future generations effect $FG(s)$ discussed below is absent. We know that $PE(s^{CE}) + CG(s^{CE}) < 0$, and thus the Ramsey government will choose a saving rate that is smaller than the competitive equilibrium saving rate. Since s^{CE} is associated with a zero tax on capital, Proposition 1 implies that in the absence of future generations the Ramsey government will unambiguously tax capital at a positive rate. Davila et al. (2012) derive a similar result in a closely related two-period model. In Section 4.3 we will state the explicit optimal tax formula for the case $\theta = 0$.

3. The effect of a higher saving rate today on future generations through a higher capital

stock from tomorrow on, $k'(s)$, is encoded in the term

$$FG(s) = \theta W'(k'(s)) \frac{dk'(s)}{ds} = (1 - \kappa)(1 - \alpha)k^\alpha \theta W'(k'(s))$$

and depends on the relative social welfare weights of future generations $\theta = \frac{\omega_{t+\tau}}{\omega_t}$. When $W'(k'(s)) > 0$, future generations benefit from being born with a higher capital stock, and the Ramsey government might choose a higher saving rate than emerging in competitive equilibrium where households do not internalize the pecuniary externality on future generations induced by their savings choice. As a result, the Ramsey government might find it optimal to subsidize capital if θ is sufficiently large, in contrast to the case where future generations are not valued at all and $\theta = 0$. In general, it is difficult to sign $FG(s)$. While future generations unambiguously benefit from a higher capital stock in their first period of life, a higher capital stock may also lead them to increase their saving rate if the IES exceeds one which increases the capital stock in the second period of life, thereby increasing labor income risk and potentially rendering $FG(s)$ negative.

4.2 Explicit Solution of the Ramsey Tax Problem

We now provide a complete analytical characterization of the Ramsey optimal tax problem under the assumption 2 that utility is logarithmic. As in the standard neoclassical growth model, the recursive Ramsey problem with log-utility has a unique closed-form solution, which can be obtained by the method of undetermined coefficients, see Appendix B:

Proposition 2. *Suppose assumptions 1, 2 and 3 are satisfied. Then the solution of the Ramsey problem is characterized by a constant saving rate*

$$s_t = s^* = \frac{\alpha(\beta + \theta)}{1 + \alpha\beta} \quad (19)$$

and a sequence of capital stocks that satisfy

$$k_{t+1} = s^*(1 - \kappa)(1 - \alpha)k_t^\alpha$$

with initial condition k_0 . The associated value function and its derivative are given by

$$\begin{aligned} W(k) &= \Theta_0 + \frac{\alpha(1 + \alpha\beta)}{(1 - \alpha\theta)} \ln(k) \\ W'(k) &= \frac{\alpha(1 + \alpha\beta)}{(1 - \alpha\theta)k}. \end{aligned}$$

The Ramsey allocation is implemented with a constant capital tax $\tau = \tau(\beta, \theta, \kappa, \alpha; \Psi)$

$$1 - \tau = \frac{(\theta + \beta)}{(1 - \alpha\theta) \beta \Gamma(\alpha, \kappa; \Psi)}, \quad (20)$$

where Γ is a positive constant defined in equation (9) and just depends on parameters.¹³

Corollary 3. *The optimal saving rate is independent of the extent of income risk and strictly increasing in the social discount factor θ and the individual discount factor β .*

Corollary 4. *The optimal capital tax rate is strictly increasing in income risk Γ , strictly decreasing in θ , strictly increasing in β and strictly decreasing in the labor income share κ of the old.*

It is noteworthy that not only is the optimal saving rate constant and does not depend on the level of the capital stock, but it also is independent of the extent of income risk η . This is true despite the fact that for a *given* tax policy higher income risk induces a higher individually optimal saving rate, as shown in Section 3.3. The Ramsey government finds it optimal to implement a capital tax that is increasing in the amount of income risk, exactly offsetting the partial equilibrium incentive to save more as income risk increases.

One advantage of the complete characterization of the recursive problem, relative to the sequential formulation in Appendix B, is that we can give a clean decomposition of the

¹³Appendix B shows, using the sequential formulation of the problem, that for arbitrary welfare weights the optimal saving rate is still independent of the capital stock and given by

$$s_t = \frac{1}{1 + \left(\alpha\beta + \alpha(1 + \alpha\beta) \sum_{j=1}^{\infty} \frac{\omega_{t+j}}{\omega_t} \alpha^{j-1} \right)^{-1}}.$$

The saving rate in the proposition is a special case under the assumption $\frac{\omega_{t+1}}{\omega_t} = \theta$ for all t .

three forces determining the optimal Ramsey saving rate. We now find that

$$\begin{aligned} PE(s) &= \frac{-1}{(1-s)} + \frac{\alpha\beta}{s} \Gamma(\alpha, \kappa; \Psi) \\ CG(s) &= \frac{\alpha\beta}{s} [1 - \Gamma(\alpha, \kappa; \Psi)] \\ FG(s) &= \frac{\theta\alpha(1 + \alpha\beta)}{(1 - \alpha\theta)s}, \end{aligned}$$

where we note that

$$\Gamma(\alpha, \kappa; \Psi) > \frac{1}{\kappa(1 - \alpha) + \alpha} \geq 1.$$

The first inequality is strict as long as Ψ is nondegenerate and $\kappa > 0$, and the second inequality is strict as long as $\kappa < 1$. Thus $[1 - \Gamma(\alpha, \kappa; \Psi)] \leq 0$, with strict inequality if $\kappa < 1$. This implies

$$PE(s) \gtrless 0, CG(s) < 0, FG(s) > 0.$$

Recall that the saving rate s^{CE} in the competitive equilibrium with zero taxes satisfies $PE(s^{CE}) = 0$. Thus, starting from zero taxes, the only reason to tax capital is the current generations general equilibrium effect which unambiguously lowers the desired saving rate and pushes the tax rate up above zero. Against this works the future generations effect with size controlled by θ that calls for a higher saving rate and thus a negative tax rate. Finally note that

$$\begin{aligned} PE(s) + CG(s) &= \frac{-1}{(1-s)} + \frac{\alpha\beta}{s} \Gamma(\alpha, \kappa; \Psi) + \frac{\alpha\beta}{s} [1 - \Gamma(\alpha, \kappa; \Psi)] \\ &= \frac{-1}{(1-s)} + \frac{\alpha\beta}{s} \end{aligned} \tag{21}$$

and thus the partial equilibrium incentive to save more when income risk rises is *exactly* cancelled out by the general equilibrium effect on factor prices. Thus the simple solution with log-utility of the Ramsey problem masks the presence of a partial equilibrium and a general equilibrium effect, of which the risk terms turn out to exactly cancel each other out.

4.3 Discussion of Optimal Tax Rates

In this section we use the sharp characterization of optimal Ramsey saving rates and capital taxes from equation (20) to discuss further properties of the optimal Ramsey capital tax

rates. The following proposition, which follows immediately from inspection of (20), gives conditions under which the optimal Ramsey capital tax is positive, and, in contrast, conditions under which capital is subsidized. For the next proposition, recall that for $\theta = 0$ only the utility of the first generation receives weight in the social welfare function, whereas $\theta = 1$ amounts to the Ramsey government maximizing steady state welfare.

Proposition 3. *There is a threshold social discount factor $\bar{\theta}$ such that for all $\theta \geq \bar{\theta}$ capital is subsidized whereas for all $\theta < \bar{\theta}$ it is taxed in every period.¹⁴ This threshold is given as*

$$\bar{\theta} = \frac{(\Gamma - 1) \beta}{1 + \alpha \beta \Gamma} > 0.$$

If $\bar{\theta} \geq 1$, capital is taxed even when the Ramsey government maximizes steady state welfare. If $\bar{\theta} < 1$ the Ramsey government maximizing steady state welfare should subsidize capital.

Note that these results also apply to the model without income risk, which provides a useful benchmark to interpret the general findings. Without risk and a Ramsey government that maximizes steady state welfare ($\theta = 1$) the optimal capital tax is given by

$$\tau = 1 - \frac{(1/\beta + 1)(1 - (1 - \kappa)(1 - \alpha))}{(1 - \alpha)}.$$

In Appendix C we show that in this case $\tau < 0$ if and only if the competitive equilibrium without taxes is dynamically efficient (i.e. has an interest rate $R > 1$, or equivalently, a capital stock below the golden rule capital stock k^{GR}). This suggests the possibility that without income risk the competitive economy is dynamically efficient and the government optimally *subsidizes* capital in the steady state, but with sufficiently large income risk the result reverses and the Ramsey government finds it optimal to tax capital in the steady state. The following proposition, again proved in Appendix C, shows that this is indeed the case.

Proposition 4. *Let $\theta = 1$ such that the Ramsey government maximizes steady state welfare, and denote by s^* the associated optimal saving rate. Furthermore denote by s^{CE} the steady*

¹⁴If $\theta = 0$ then we can, by inserting the private Euler equation in the $PE(s)$ effect, directly derive the optimal tax rate on capital for an arbitrary strictly concave and differentiable utility function and a CRTS production function $f(k, 1)$ with strictly positive and strictly decreasing marginal products, as

$$\tau = \frac{-f_{lk}(k'(s))}{f_k(k'(s))} \times \frac{E[u'(c(\eta))(\kappa\eta - 1)]}{E[u'(c(\eta))]} > 0.$$

Note that although this result establishes that the optimal capital tax rate is positive in the two period model ($\theta = 0$), it does not give the optimal tax rate in closed form since consumption $c(\eta)$ is endogenous.

state equilibrium saving rate in the absence of government policy and by s^{GR} the golden rule saving rate that maximizes steady state aggregate consumption. Finally assume that the private discount factor β is sufficiently small.¹⁵

1. Let income risk be large: $\Gamma > \frac{1}{\beta[(1-\alpha)-1/\bar{\Gamma}]}$. Then the steady state competitive equilibrium is dynamically inefficient, $s^{GR} < s^{CE}$, and $s^* < s^{CE}$, and the optimal capital tax rate has $\tau > 0$.

2. Let income risk be intermediate:

$$\Gamma \in \left(\frac{1+\beta}{(1-\alpha)\beta}, \frac{1}{[(1-\alpha)-1/\bar{\Gamma}]\beta} \right).$$

Then the steady state competitive equilibrium is dynamically efficient, $s^* < s^{CE} < s^{GR}$, but optimal capital taxes are nevertheless positive, $\tau > 0$.

3. Let income risk be small:

$$\Gamma \in \left[\bar{\Gamma}, \frac{1+\beta}{(1-\alpha)\beta} \right)$$

Then the steady state competitive equilibrium is dynamically efficient, $s^{CE} < s^{GR}$, and $s^{CE} < s^*$, and optimal capital taxes are negative.

The interesting result is case 2: in the presence of income risk the Ramsey government maximizing steady state welfare might want to tax capital even though this reduces aggregate consumption (since the equilibrium capital stock is not inefficiently high) because of the *CG* effect: a lower capital stock shifts away income from risky labor income to non-risky capital income, and for moderate income risk this effect dominates the future generations effect as parametrized by θ . Note that the bounds in the previous proposition can of course be directly defined in terms of the variance of the idiosyncratic income shock η , to a second order approximation of the integral defining Γ (see Appendix G.2).

¹⁵Formally, assume that $\beta < [(1-\alpha)\bar{\Gamma}-1]^{-1}$. If this condition is violated, then the steady state competitive equilibrium is dynamically inefficient and only case 1 below is relevant, that is, the optimal capital tax rate is positive for all degrees of income risk.

4.4 Capital Stock Dynamics and Capital *Income* Taxes

The discussion in the previous section concerned the optimal, time-invariant saving rate. The saving rate, together with the law of motion for the capital stock

$$k_{t+1} = s_t(1 - \kappa)(1 - \alpha)k_t^\alpha = \frac{\alpha(\theta + \beta)(1 - \kappa)(1 - \alpha)}{1 + \alpha\beta} k_t^\alpha$$

and the initial condition k_0 determine the entire time path for the capital stock. That sequence $\{k_t\}_{t=1}^\infty$ is independent of the amount of income risk and converges monotonically to the steady state

$$k^* = \left[\frac{\alpha(\theta + \beta)(1 - \kappa)(1 - \alpha)}{1 + \alpha\beta} \right]^{\frac{1}{1-\alpha}},$$

either from above if $k_0 > k^*$ or from below, if $k_0 < k^*$. Again, the optimal tax policy that implements this allocation depends on the extent of income risk, as shown above. We can now also make precise the relation between the capital taxes τ_t studied thus far, and the implied optimal capital *income* taxes τ_t^k . These are related by the equation¹⁶

$$1 + (R_t - 1)(1 - \tau_t^k) = R_t(1 - \tau_t)$$

and thus

$$\tau_t^k = \frac{R_t}{R_t - 1} \tau_t,$$

where the gross return is given by $R_t = \alpha(k_t)^{\alpha-1}$. As long as $R_t > 1$ for all t , capital taxes and capital income taxes have the same sign. A sufficient condition for this is

Assumption 4. *The initial capital stock and the model parameters satisfy $k_0 < \alpha^{\frac{1}{1-\alpha}}$ and*

$$\frac{1 + \alpha\beta}{(\theta + \beta)(1 - \kappa)(1 - \alpha)} > 1.$$

This assumption assures that net returns are strictly positive at all times in the Ramsey equilibrium, since $R_0 = \alpha(k_0)^{\alpha-1} > 1$ and $R^* = \alpha(k^*)^{\alpha-1} > 1$, (and because the sequence of R_t along the transition is monotone) and thus the Ramsey allocation can be

¹⁶This equation assumes that the government does not permit the expensing of investment from capital income taxes. Abel (2007) shows that if such expensing is allowed, capital income taxes are nondistortionary (under appropriate ancillary assumptions). Since the Ramsey government optimally distorts the capital accumulation decision of private households in this paper, one implication of our results is that it is not optimal for the government to permit full expensing of investment in our environment (under the maintained other restrictions on the tax instruments).

supported by capital income taxes of the same sign as the corresponding wealth taxes. Under assumption 4 therefore all interpretations and qualitative results extend without change to capital income taxes.

4.5 Robustness to Alternative Assumptions

We now briefly comment on the robustness of our main findings to alternative modeling assumptions, considering in turn ex-ante household heterogeneity in labor productivity, stochastic investment returns, time varying technological progress and population growth and, finally, endogenous labor supply.

4.5.1 Ex-Ante Heterogeneity in Labor Productivity

Assume that households differ according to permanent labor productivity ν , so that labor productivity of type ν is given by $\nu(1 - \kappa)$ when young and $\kappa\nu\eta$ when old. Further assume that the distribution of second period shocks η is independent of permanent productivity type ν and that the cross-sectional distribution of ν has mean 1. The government continues to tax capital at a uniform rate and rebates revenues lump-sum within each ν -type according to the groups' tax payments. In Appendix D.1 we show that, perhaps not surprisingly given homotheticity of preferences, the general equilibrium saving rate is identical across all ν -types households, and continues to be given by equation (12) from the benchmark model. Also, the optimal saving rate s implemented by the utilitarian (across ν) Ramsey government remains unchanged, see equation (19), and so is the implementing optimal tax rate (20). Households that differ in permanent productivity are just scaled-up versions of each other so that with homothetic preferences they will choose the same saving rate in general equilibrium. The Ramsey planner has to respect this choice and thus implements the same capital tax policy as before.

4.5.2 Idiosyncratic Return Risk

Return to the model with ex ante identical households, but now also consider idiosyncratic return shocks, denoted by ϱ . After-tax gross returns in the second period of life are now stochastic and given by $R_{t+1}\varrho_{t+1}(1 - \tau_{t+1})$, and return risk and labor income risk η_{t+1} may be correlated. We further assume that transfer payments are contingent on the rate of return realization, $T_{t+1}(\varrho) = a_{t+1}R_{t+1}\varrho_{t+1}\tau_{t+1}$.

Given these assumptions, we show in Appendix D.2 that all our results go through completely unchanged, with the impact of idiosyncratic risk now expressed in terms of the random variable $\delta_{t+1} = \frac{\eta_{t+1}}{\varrho_{t+1}}$ and its associated distribution Π . Thus, the constant $\Gamma = \Gamma(\alpha, \kappa; \Pi)$ reflecting risk in (12) is now parameterized by Π . An increase in labor income risk still increases the saving rate, and an increase in returns risk decreases it as saving becomes less attractive. The optimal Ramsey saving rate continues to be given by equation (19), implemented with a tax rate according to (20), with the constant Γ now dependent on the distribution Π .

4.5.3 Time Varying Productivity and Population Growth

Our results fully extend to a model with deterministic technological progress. Assume that production is given by $Y_t = K_t^\alpha (A_t L_t)^{1-\alpha}$ where TFP A_t evolves according to $A_t = (1 + g_t)A_{t-1}$ at a time varying rate g_{t+1} . This time-varying growth rate cancels out in the household optimization problem and adds maximization irrelevant additive terms to the Ramsey problem, leaving the optimal saving rate and tax rate on capital that implements it completely unchanged.

In contrast, positive population growth at a constant rate n impacts the general equilibrium by reducing the capital-labor ratio, therefore lowering wages and increasing asset returns faced by cohort t in period $t + 1$ thereby increasing the competitive equilibrium saving rate for given taxes. The optimal Ramsey saving rate, however, is not affected by population growth, but the optimal tax on capital implementing that rate is.¹⁷ Finally, in the steady state of the model, positive population growth shrinks, but does not eliminate, the size of the intermediate risk interval characterized in Proposition 4 for which the economy is dynamically efficient yet capital is taxed at a positive rate, and the reform towards that tax rate is Pareto improving.

4.5.4 Endogenous Labor Supply and Labor Income Taxation

Thus far we have assumed that labor supply is inelastic, to focus on the role of precautionary saving for capital (income) taxation. Furthermore, we have assumed that the Ramsey government cannot complete markets by redistributing intra-generationally, either by pro-

¹⁷If the population growth rate is time varying, then the general equilibrium saving rate is time varying as well. With a Utilitarian objective we can then no longer characterize the solution to the Ramsey problem in closed form. If the planner instead maximizes discounted per capita utilities, then our closed form results go through unchanged and the optimal Ramsey saving rate continues to be a constant.

gressive taxation or income-contingent transfers. If it could, then the Ramsey government, in the presence of exogenous labor supply, would trivially provide full insurance against η -risk by taxing all labor income in the second period at 100% and rebating it in a lump-sum fashion among all households. Consequently there is no role for taxes on capital in such an environment since the precautionary saving mechanism we have highlighted disappears.

Of course, with endogenous labor supply taxing labor income at a confiscatory rate is no longer optimal, which raises the question how the government should then tax labor and capital income in a world with idiosyncratic income risk and overlapping generations. We address this question in Krueger, Ludwig, and Villalvazo (2019), by endogenizing labor supply in the second period of life, interpreting this choice as decision how long to work and when to retire. The government taxes labor income and capital at potentially different but linear rates, with tax proceeds rebated in a lump-sum fashion to all members of the same generation. Since transfers are not contingent on η -shock realizations, this scheme provides some insurance against η -risk. We find that the optimal Ramsey saving rate continues to be constant, as is aggregate labor supply, and the optimal capital income and labor tax rates implementing these allocations are constant as well (and non-zero almost surely).¹⁸ As in this paper optimal taxes are increasing in labor productivity risk, and positive as long as that risk is sufficiently large.

5 Efficiency Properties of the Ramsey Equilibrium

In this section we discuss the welfare properties of the Ramsey equilibrium characterized thus far. By construction, the Ramsey allocation is the best allocation, given the weights in the social welfare function, that a government that needs to respect equilibrium behavior of households and is restricted to proportional taxes on capital can implement. We establish two main results. First, defining constrained efficient allocations as those chosen by a social planner that cannot directly transfer consumption across households of different ages and with different idiosyncratic shocks (as in Davila et al. (2012)), we show that the Ramsey equilibrium is constrained efficient in this precise sense. Second, we prove that if the optimal Ramsey saving rate that maximizes steady state welfare $s^*(\theta = 1)$ is smaller than the steady state saving rate in the competitive equilibrium without government s^{CE} , then implementing $s^*(\theta = 1)$ through positive capital taxes yields a Pareto-improving transition

¹⁸Under the assumptions maintained in the previous section, notably log-utility, geometrically declining social welfare weights, as well as balanced growth preferences on labor.

from the initial laissez-faire steady state towards the steady state associated with $s^*(\theta = 1)$. This is true even if the steady state equilibrium is dynamically efficient.

5.1 Constrained Efficiency of Ramsey Equilibria

The Ramsey government cannot implement fully Pareto efficient allocations, characterized in Appendix E.1. We now ask whether the government can at least achieve constrained efficiency with the set of instruments it has. A constrained efficient allocation is an allocation of capital and consumption that maximizes social welfare subject to the constraint that the allocation does not permit transfers across old households with different η realizations.

Define the set of allocations that are feasible for the *constrained* planner as

$$c_t^y + \int c_t^o(\eta_t) d\Psi + k_{t+1} = k_t^\alpha \quad (22a)$$

$$c_t^o(\eta_t) = k_t MPK(k_t) + \kappa \eta_t MPL(k_t). \quad (22b)$$

The first constraint is simply the resource constraint. The second constraint restricts transfers across different η households: old age consumption is required to equal capital income plus an η household's share of labor income, where the returns to capital and labor are equal to the factors' relative productivities. The constrained planner might find it optimal, however, to manipulate factor prices by choosing a different sequence of capital stocks, relative to that of a competitive equilibrium (without or with tax policy). Note that these constraints also imply that

$$\int c_t^o(\eta_t) d\Psi = k_t MPK(k_t) + \kappa MPL(k_t) \quad (23a)$$

$$c_t^y = (1 - \kappa) MPL(k_t) - k_{t+1} \quad (23b)$$

so that no *intergenerational* transfers are permitted either, relative to the competitive equilibrium. A constrained efficient allocation is one that maximizes societal welfare $W = \sum_{t=-1}^{\infty} \omega_t V_t$ subject to (22a) and (22b). The question is whether the simple tax policy we consider is sufficient to offset the precautionary savings externality on factor prices, and implement the constrained efficient allocation. The answer is yes, as the following proposition (proved in Appendix E.2) shows.¹⁹

¹⁹With ex-ante heterogeneity of the form analyzed in Section 4.5.1, the Ramsey planner can no longer implement the constrained efficient outcome, because the constrained planner can achieve some redistribution through mandating saving rates that differ across productivity type whereas the Ramsey planner cannot. A

Proposition 5. *The Ramsey allocation is constrained-efficient.*²⁰

5.2 Pareto-Improving Tax Reforms

In this section we show that starting from the *steady state competitive equilibrium without taxes* as initial condition, switching to the Ramsey optimal savings and tax policy that maximizes steady state welfare yields a Pareto improvement, in that all generations, including those along the transition, are better off. This is true as long as the optimal Ramsey saving rate is smaller than the steady state competitive equilibrium saving rate, and is true even if the original competitive steady state equilibrium is dynamically efficient in the sense of satisfying $k_0 < k^{GR}$ (and thus $R_0 > 1$), where k^{GR} is the golden rule capital stock maximizing steady state aggregate consumption and k_0 is the initial steady state equilibrium capital stock.

Proposition 6. *Let s^{CE} denote the saving rate in a steady state competitive equilibrium with zero taxes. Assume that $s^{CE} > s^*$. Then a government policy that sets $\tau_t = \tau^* > 0$ leads to a Pareto improving transition from the initial steady state with capital k_0 towards the new steady state associated with tax policy τ^* .*

The proof of this proposition in Appendix E.3 shows that *all generations* benefit from the government implementing a saving rate that is lower than the initial competitive equilibrium rate despite the fact that it lowers the capital stock, thus aggregate production, wages and consumption along the transition. Utility gains arise from higher consumption when young due to the lower saving rate. Since along the transition $c_t^y = (1 - s^*)(1 - \kappa)(1 - \alpha)k_t^\alpha$ and since the capital stock is decreasing along the transition, utility gains are highest in

similar result applies in infinite horizon Aiyagari-style models, as emphasized by Davila et al. (2012).

²⁰This result relates our analysis to the Mirrleesian tradition to optimal capital income taxation, see e.g. Golosov et al. (2003) and Farhi and Werning (2012). Consider a Mirrleesian planner who chooses optimal allocations under the constraint that η -shocks are private information of households. Assume that this Mirrleesian planner additionally is not permitted to use *intergenerational* transfers, i.e. impose the constraints (23). The Mirrleesian planner would want to implement transfers across $\hat{\eta}$ -types (where $\hat{\eta}$ denotes the productivity reports of households), and provide insurance against low $\hat{\eta}$ realizations. The resource constraint and assumed absence of intergenerational transfers implies that these transfers have to net out to zero in every period. However, under such a transfer scheme, all high- η households have an incentive to report low $\hat{\eta}$ and therefore any transfer scheme across $\hat{\eta}$ -types is not incentive compatible. Furthermore the Mirrleesian planner has no other means to incentivize truthful reporting (e.g. by making future consumption or labor supply contingent on the $\hat{\eta}$ reports). Thus, transfers across $\hat{\eta}$ -households are infeasible and the constraint (22b) would result as a consequence of incentive compatibility in the Mirrleesian problem. The Mirrleesian planner would therefore implement, somewhat trivially under the maintained assumptions, the constrained efficient allocation of Section 5.1, which coincides with the Ramsey optimum shown in the proposition.

the first period of the transition and monotonically decreasing along the transition. In contrast, utility losses emerge from lower consumption when old, which in general equilibrium is $c_{t+1}^o = [\alpha + \kappa\eta_{t+1}(1 - \alpha)] k_{t+1}^\alpha$, and thus is monotonically decreasing along the transition. Thus, the net welfare gains are highest for the initial generations and monotonically declining along the transition. But by the choice of s^* the government insures that generations in the new steady state benefit from the reform, and the monotonicity of the net welfare gains along the transition insures that all generations living through the transition are better off from implementing s^* through positive taxes on capital, $\tau^* > 0$.

The result in the previous proposition is of course not surprising if s^{CE} is larger than the golden rule implementing saving rate s^{GR} and the initial steady state competitive equilibrium is dynamically inefficient to start with. However, for intermediate risk, i.e., for

$$\Gamma \in \left(\frac{1 + \beta}{(1 - \alpha)\beta}, \frac{1}{[(1 - \alpha) - 1/\bar{\Gamma}]\beta} \right)$$

Proposition 4 shows that $s^* < s^{CE} < s^{GR}$, and thus the steady state equilibrium is dynamically efficient. Proposition 6 establishes that setting $\tau^* > 0$ implements a Pareto-improving transition even in this case.

Note that Proposition 6 discusses a potentially massive permanent policy reform from $\tau = 0$ to $\tau = \tau^*$. A reform decreasing the saving rate s^{CE} marginally but permanently by implementing a marginal tax hike $\tau = \varepsilon > 0$ also leads to a Pareto improvement, under exactly the same conditions as in the previous proposition.

Also observe that the preceding arguments, and thus our results on Pareto improving transitions by either implementing the optimal long-run Ramsey saving rate $s^* < s^{CE}$ or by a marginal reduction of the saving rate from s^{CE} hold for arbitrary additively separable lifetime utility whenever the period utility function $u(\cdot)$ is strictly increasing. All we require for the result is an initial laissez-faire equilibrium allocation featuring $s^* < s^{CE} < s^{GR}$, the exact conditions for this inequality to be satisfied of course depends on the preference structure. Also note that, in general, the tax rate on capital required to implement the Pareto-improving time-constant saving rate will be time-varying, rather than constant, as in the logarithmic case.

Observe that the argument above on the sources of these net utility gains along the transition—higher, and monotonically decreasing, consumption when young and lower, and monotonically decreasing, consumption when old—do not rest on the presence or extent of income risk. However, whether the initial laissez-faire equilibrium satisfies the

inequalities $s^* < s^{CE} < s^{GR}$ of course depends on risk.

Finally, it is important to note that the converse of Proposition 6 is not true: even if $s^{CE} < s^*$, implementing the Ramsey optimal steady state ($\theta = 1$) savings subsidy $\tau^* < 0$ and associated higher saving rate s^* does *not* lead to a Pareto improving transition. Appendix E.4 shows that the generation born into the first period of the hypothetical policy-induced transition loses from this policy change. In fact, not only is implementing $\tau^* < 0$ not Pareto improving if $s^{CE} < s^*$, any (marginal) policy reform that induces a period 1 saving rate above the competitive saving rate with zero taxes, s^{CE} , does not result in a Pareto improvement, since it makes the first generation strictly worse off.

6 General Intertemporal Elasticity of Substitution ρ and Risk Aversion σ

In this section we extend our results to a more general utility function with intertemporal elasticity of substitution ρ and risk aversion σ , as in Epstein and Zin (1989, 1991) and Weil (1989). While most of this analysis focusses on *steady states*, we establish that our closed form results for the *transition* go through unchanged for an IES $\rho = 1$. All details of formal derivations are relegated to Appendix F.

We now consider a utility function of the form

$$V_t = u(c_t^y) + \beta u(v(c_{t+1}^o)) \quad (24)$$

where the period utility function is

$$u(x) = \begin{cases} \frac{1}{1-\frac{1}{\rho}} x^{1-\frac{1}{\rho}} & \text{for } \rho \neq 1 \\ \ln(x) & \text{for } \rho = 1 \end{cases}$$

for $x \in \{c_t^y, v(c_{t+1}^o)\}$, and the certainty equivalent is given by

$$v(c_{t+1}^o) = \begin{cases} \left(\int c_{t+1}^o(\eta)^{1-\sigma} d\Psi(\eta) \right)^{\frac{1}{1-\sigma}} & \text{for } \sigma \neq 1 \\ \exp \left(\int \ln(c_{t+1}^o(\eta)) d\Psi(\eta) \right) & \text{for } \sigma = 1, \end{cases}$$

This preference specification was first introduced into the literature by Selden (1978, 1979).

The parameter ρ measures the IES and the parameter σ governs risk aversion.²¹ If $\sigma = \frac{1}{\rho}$ then the utility function takes the standard CRRA form.

As in Section 4, equation (13), we can write lifetime utility of a generation born in period t , in general equilibrium, as a function of the beginning of the period capital stock k_t and the saving rate s_t chosen by the Ramsey government and implemented by the appropriate choice of the capital tax τ_{t+1} . In addition, in the steady state the saving rate and the associated capital stock are related by:

$$k = ((1 - \kappa)(1 - \alpha)s)^{\frac{1}{1-\alpha}}.$$

In Appendix F we show that the objective function of the Ramsey government boils down to maximizing, by choice of the steady state saving rate, steady state lifetime utility:

$$V(s) = \tilde{\phi} \left((1 - s)^{(1-\frac{1}{\rho})} + \beta \tilde{\zeta} \tilde{\Gamma}_2 \right) s^{\frac{\alpha(1-\frac{1}{\rho})}{1-\alpha}} \quad (25)$$

where $\tilde{\phi}$ and $\tilde{\zeta} > 0$ and $\tilde{\Gamma}_2 > 0$ are constants that depend on parameter values. We find that the optimal steady state saving rate is defined implicitly as

$$s^* = \frac{\alpha}{1 - \alpha} \left[(1 - s^*) + \beta \tilde{\zeta} \tilde{\Gamma}_2 (1 - s^*)^{\frac{1}{\rho}} \right]. \quad (26)$$

From inspection of equation (26) we obtain (see Appendix F):

Proposition 7. *Suppose that $\theta = 1$ and thus the Ramsey government maximizes steady state welfare. There exists a unique optimal Ramsey saving rate $s^* \in (0, 1)$ solving equation (26). This saving rate can be implemented with a capital tax rate τ^* determined by the*

²¹This specification of Epstein-Zin-Weil preferences is also used by other papers in the literature, e.g., in Bommier et al. (2017). Note that V_t represents the same ordinal ranking over current consumption c_t^y and the certainty equivalent over future risky consumption $c_{t+1}^o(\eta_{t+1})$ as the more commonly used specification

$$\tilde{V}_t = \left\{ (1 - \tilde{\beta})(c_t^y)^{1-\frac{1}{\rho}} + \tilde{\beta} [v(c_{t+1}^o)]^{1-\frac{1}{\rho}} \right\}^{\frac{1}{1-\frac{1}{\rho}}}$$

since one is a monotone transformation of the other, $V_t = \frac{\tilde{V}_t^{1-\frac{1}{\rho}}}{(1-\tilde{\beta})(1-\frac{1}{\rho})} - \frac{(1+\beta)}{1-\frac{1}{\rho}}$, where $\beta = \frac{\tilde{\beta}}{1-\tilde{\beta}}$. However, since the Ramsey problem is stated in terms of the weighted sum of *cardinal* utilities, a monotone transformation of the utility function will in general alter the Ramsey problem. When focusing on a steady state analysis, this concern does not arise, since the same saving rate (and associated tax rate) maximizes steady state V and its monotone transformation \tilde{V} .

competitive equilibrium Euler equation:

$$1 = (1 - \tau^*)\alpha\beta((1 - \kappa)(1 - \alpha))^{(\frac{1}{\rho}-1)} \frac{(1 - s^*)^{\frac{1}{\rho}}}{s^*} \tilde{\Gamma}. \quad (27)$$

Note that all comparative statics results, especially those with respect to an increase in income risk, can be deduced from an analysis of equations (26, 27). Income risk affects the optimal Ramsey savings rate s^* and associated implementing tax rate τ^* only through the constants $\tilde{\Gamma}$, $\tilde{\Gamma}_2$ which are given as:

$$\tilde{\Gamma} = v^{(\sigma - \frac{1}{\rho})} \Gamma \quad (28a)$$

$$\tilde{\Gamma}_2 = v^{(1 - \frac{1}{\rho})}, \quad (28b)$$

where we had defined the constant Γ above for the log-case, and is now given by:

$$\Gamma = \int (\kappa\eta_{t+1}(1 - \alpha) + \alpha)^{-\sigma} d\Psi(\eta_{t+1}) \quad (29)$$

and where the constant v is the certainty equivalent of η defined as

$$v = \begin{cases} [\int (\alpha + (1 - \alpha)\kappa\eta)^{1-\sigma} d\Psi(\eta)]^{\frac{1}{1-\sigma}} & \text{for } \sigma = 1 \\ \exp(\int \ln(\alpha + (1 - \alpha)\kappa\eta) d\Psi(\eta)) & \text{otherwise.} \end{cases} \quad (30)$$

In Appendix G.1 we prove the following result relating the extent of income risk to the constants $\tilde{\Gamma}$, $\tilde{\Gamma}_2$ which are in turn crucial for the comparative statics results in Section 6.2.

Lemma 1. *An increase in income risk (a mean-preserving spread of η) reduces v and increases $\tilde{\Gamma}_2$ if and only if $\rho \leq 1$ and increases $\tilde{\Gamma}$ if $\rho \leq 1$, or $\rho > 1$ and $\sigma < 1/\rho$.*

Note that the condition that characterizes the relation between income risk and $\tilde{\Gamma}_2$ is necessary and sufficient whereas the two alternative conditions that characterize the relation between income risk and $\tilde{\Gamma}$ are only sufficient. The dependency of precautionary savings on both risk aversion and the IES with recursive preferences was demonstrated by Kimball and Weil (2009), and the sufficient conditions provided in the Lemma are stated in their Propositions 5 and 6. We provide further intuition for this result below when discussing implementation of the optimal Ramsey policy.

6.1 Unit Elasticity of Substitution $\rho = 1$

Recognizing that for an IES of $\rho = 1$ we have $\tilde{\zeta} = \tilde{\Gamma}_2 = 1$, direct calculations yield:

Proposition 8. *Suppose that $\rho = 1$. Then the solution of the Ramsey problem is the same as for log-utility in Section 4. That is, the optimal, constant saving rate is given by*

$$s = \frac{\alpha(\beta + \theta)}{1 + \alpha\beta}.$$

The tax rate τ implementing this saving rate as a competitive equilibrium is given by

$$1 = (1 - \tau) \left(\frac{1 - s}{s} \right) \alpha\beta\tilde{\Gamma}$$

and thus is strictly increasing in income risk measured by $\tilde{\Gamma}$.

Note that the optimal Ramsey saving rate does neither depend on income risk nor on risk aversion, but that the optimal capital tax rate τ implementing this saving rate is increasing in income risk, and *does* depend on risk aversion through the constant $\tilde{\Gamma}$. Also note that although here we state this result for steady states only, Appendix F.2 shows that the entire analysis of Section 4 with log-utility (including the dynamic programming formulation and the analysis of the transition path) goes through completely unchanged (by only replacing Γ by $\tilde{\Gamma}$) for general Epstein-Zin-Weil utility as long as the IES is unity, $\rho = 1$.

6.2 The Impact of Risk on the Optimal Saving and Tax Rate: Disentangling Risk Aversion and IES: $\frac{1}{\sigma} \neq \rho \neq 1$

Thus far, we have demonstrated that an IES of 1 is sufficient (and, as turns out, necessary) for the result that the optimal Ramsey saving rate can be solved in closed form, is constant over time and independent of the extent of income risk. In this section we investigate how income risk impacts the optimal Ramsey saving and capital tax rate for general IES and risk aversion (ρ, σ) where CRRA utility is a special case $\rho = 1/\sigma$. From equation (26) we immediately observe that the optimal steady state saving rate s is strictly increasing in the constant $\tilde{\Gamma}_2$ summarizing the impact of income risk. The response of s to income risk immediately follows from the impact of an increase in risk on $\tilde{\Gamma}_2$ in Lemma 1. We have

Proposition 9. *An increase in income risk increases the optimal steady state Ramsey saving rate s^* if and only if $\rho < 1$ and decreases it if and only if $\rho > 1$.*

Thus the direction of the change in s with respect to income risk is exclusively determined by the IES ρ , with the log-case acting as a watershed. Of course how strongly the saving rate responds to an increase in income risk is also controlled by risk aversion through the term $\tilde{\Gamma}_2$. What is the intuition for this result? Suppose the economy is in the steady state associated with a given extent of income risk and the optimal Ramsey tax policy, and now consider an increase in income risk. The Ramsey government can always neutralize the response of private households' savings behavior, by appropriate adjustment of the tax rate on capital to implement the new optimal saving rate.²²

The question is then how the saving rate desired by the Ramsey government itself changes. Households (and thus the Ramsey government) obtain utility from safe consumption when young and risky consumption when old, and the desire for smoothing utility from safe consumption when young and the certainty equivalent of consumption when old is determined by the IES ρ . As risk increases, old age consumption is now a less effective way to generate utility, and the certainty equivalent of old-age consumption declines, holding the consumption allocation constant. Whether the Ramsey government wants to raise or lower old-age consumption (by increasing or reducing the saving rate) depends on how much households value a smooth life cycle utility profile. In the log-case the two forces exactly balance out and the Ramsey saving rate does not respond to income risk at all. In contrast, if households strongly desire a smooth path of (the certainty equivalence of) consumption, then the Ramsey government compensates for the loss of old-age certainty equivalent consumption from larger income risk by saving at a higher rate, and s increases with income risk if the IES ρ is small. The reverse is true for a high IES.

Finally, we can also determine the impact of income risk on optimal steady state capital taxes. From equation (27) the optimal Ramsey tax rate is given by

$$1 = (1 - \tau^*)\alpha\beta((1 - \kappa)(1 - \alpha))^{(\frac{1}{\rho}-1)} \frac{(1 - s^*)^{\frac{1}{\rho}}}{s^*} \tilde{\Gamma}. \quad (31)$$

We observe that income risk affects the optimal tax rate in two ways. First, for a given target saving rate s^* , the direct impact of income risk depends on how $\tilde{\Gamma}$ (and thus the private saving rate) responds to an increase in risk. Second, a change in income risk changes

²²We saw this explicitly in the decomposition of the first order condition of the Ramsey government in Section 4.2, where the risk term Γ from the competitive equilibrium optimality condition dropped out because the government chooses, through taxes and the associated changes in factor prices, to exactly offset the impact of higher risk on private household savings decisions. In the logic of that section, an increase in Γ increases $PE(s)$ but reduces $CG(s)$ by precisely the same factor.

the optimal saving rate s^* through $\tilde{\Gamma}_2$, as characterized in the previous proposition. The next proposition, proved in Appendix F.4, gives sufficient conditions on the IES and risk aversion (ρ, σ) under which the optimal capital tax rate τ^* is increasing in income risk, and a necessary condition required for the tax rate to be decreasing in income risk. The proof of the first proposition exploits the fact that using (26) we can rewrite equation (31) as:

$$1 = (1 - \tau^*) \left(1 - \frac{\alpha}{s^*}\right) \frac{\tilde{\Gamma}}{\tilde{\Gamma}_2}. \quad (32)$$

Proposition 10. *If $\rho \leq 1$, then an increase in income risk increases the optimal tax rate on capital. Similarly, if $\rho > 1$ and $\sigma \leq 1/\rho$, then an increase in income risk increases the optimal tax rate on capital.*

Proposition 11. *If $\rho > 1$ and $\sigma > 1/\rho$, an increase in income risk might lead to a strict reduction in the optimal tax rate τ on capital. A necessary condition for this result is that the competitive equilibrium saving rate for given τ is strictly decreasing in income risk.*

The intuition for this last proposition is that, if $\rho > \max\{1, 1/\sigma\}$, then private households might *decrease* their saving rate too much in general equilibrium in response to an increase in income risk since they do not internalize the impact of the decline of the saving rate on the capital stock and thus on wages of future generations. For the capital tax to decrease in income risk this future generations effect has to be sufficiently strong. To see this formally, in Appendix F.5 we first derive the decomposition of the first-order condition for the optimal saving rate into the terms $PE(s)$, $CG(s)$ and $FG(s)$, for the general EZW utility function, and in Appendix F.6 we use this decomposition to write equation (32) as

$$1 = \underbrace{(1 - \tau^*) \frac{\tilde{\Gamma}}{\tilde{\Gamma}_2}}_{\text{from } PE(s)+CG(s)} - \underbrace{(1 - \tau^*) \frac{\alpha}{s^*} \frac{\tilde{\Gamma}}{\tilde{\Gamma}_2}}_{\text{from } FG(s)}.$$

Since $\tilde{\Gamma}/\tilde{\Gamma}_2$ is increasing in income risk, the optimal capital tax rate τ^* can only decrease in income risk when the last term, the future generations effect, is large. This effect calls for a tax rate that decreases with income risk since s^* is decreasing in risk for $\rho > 1$.

In the next section we characterize the optimal solution of the Ramsey tax problem numerically outside the steady state. The results demonstrate that there are indeed robust regions of the preference parameter space in which the optimal sequence of tax rates in the Ramsey equilibrium is indeed strictly decreasing in income risk, in all time periods.

6.3 Numerical Exploration of Optimal Ramsey Tax Transitions for General IES $\rho \neq 1$ and Risk Aversion σ

In the previous subsection we provided a theoretical characterization of the optimal Ramsey policy under the assumption that the government maximized steady state utility, i.e. $\theta = 1$. Since no analytical results are available *outside the steady state* unless $\rho = 1$, we now solve for the optimal Ramsey tax transition numerically. We take as initial condition the steady state capital stock in the competitive equilibrium without taxes and characterize the sequences of saving rates, capital stocks, capital tax rates as well as the lifetime utility consequences from the transition, relative to the steady state without taxes.

To implement the simulations we need to choose parameters. To exploit the dynamic programming solution of the Ramsey problem with a social discount factor $\theta < 1$, but to retain the steady state results as useful benchmark for comparison, we choose $\theta = 0.9$. Our focus is to characterize how the extent of income risk affects the optimal Ramsey tax transition, and how households' preferences towards that risk (as measured by σ) and their willingness to inter-temporally substitute (as measured by ρ) shape this transition. In the main text we focus on a parameter constellation for which changes in income risk have a potentially non-monotonic impact on the optimal tax rate in steady state. Recall that this requires large σ and ρ , together with the restriction that $\sigma > 1/\rho$. Thus, here we choose $\rho = 20$ and $\sigma = 50$; in Appendix H we also present quantitative results for lower, more commonly used values of both parameters. To vary the degree of idiosyncratic income risk we assume that η is distributed log-normally and consider four levels of risk: $\sigma_{\ln \eta} \in \{0, 0.25, 1, 2\}$, and adjust the mean $\mu_{\ln \eta}$ such that $E(\eta) = 1$ for all risk parameterizations.²³ The purpose of the simulations is to illustrate qualitative properties of the Ramsey solution when analytical results are not available, rather than making firm quantitative statements.

From the initial capital stock k_0 , assumed to be the steady state capital stock absent tax policy, the Ramsey government determines the optimal sequence of saving rates from period 1 onwards, and implements them with capital income taxes from period 2 onwards. For the various parameterizations, the initial competitive equilibrium saving rate, s^{CE} , the optimal saving rate in the long-run, s_∞^* , and the optimal long-run capital *income* tax rate, τ_∞^{k*} are shown in Table 1.²⁴ We observe that for our parameterization with a high IES the steady

²³We approximate the distribution with $n = 21$ Gaussian quadrature nodes. Other parameters include $\alpha = 0.2$, $\beta = 0.8$ and $\kappa = 0.5$. This choice of α and κ implies that the golden rule saving rate is $s^{GR} = 0.5$.

²⁴All economies considered here are dynamically efficient in that the saving rate s^{CE} is less than the golden rule saving rate $s^{GR} = 0.5$ and returns on capital are positive. Consequently, taxes on capital, and on

state competitive equilibrium saving rate s^{CE} is inverse u-shaped in income risk, initially increasing but eventually declining in $\sigma_{\ln \eta}$ when income risk exceeds some threshold.²⁵

Table 1: Saving Rates in Competitive Equilibrium and Optimal Long-Run Saving & Capital Income Tax Rates: EZW-Preferences with $\rho = 20$, $\sigma = 50$

	s^{CE}	s_{∞}^*	τ_{∞}^{k*}
$\sigma_{\ln \eta} = 0$	0.38	0.41	-0.13
$\sigma_{\ln \eta} = 0.25$	0.48	0.34	0.52
$\sigma_{\ln \eta} = 1$	0.44	0.28	0.60
$\sigma_{\ln \eta} = 2$	0.42	0.27	0.56

Notes: Saving rates in the initial competitive equilibrium, s^{CE} , and optimal long-run saving, s_{∞}^* , and capital income tax rates τ_{∞}^{k*} for $\alpha = 0.2$, $\beta = 0.8$, $\kappa = 0.5$, $\sigma_{\ln \eta} \in \{0, 0.25, 1, 2\}$, $\theta = 0.9$ and $\rho = 20$, $\sigma = 50$.

In Proposition 11 we showed that when the Ramsey government maximizes steady state utility ($\theta = 1$), a necessary condition for optimal capital taxes to *decline* with income risk is that the competitive equilibrium saving rate falls with income risk. This result is apparent in Table 1 in that s^{CE} starts to decline with income risk after $\sigma_{\ln \eta} > 0.32$, and the optimal long-run Ramsey tax rate also eventually decreases with income risk, but not until after $\sigma_{\ln \eta} > 1$. Finally, note that only in the deterministic economy with $\sigma_{\ln \eta} = 0$, the competitive equilibrium saving rate is lower than the long-run optimum saving rate; in all other economies it is higher, and optimal taxes on capital income are positive.²⁶

Figure 1 shows, in panels (a) and (b), the optimal dynamic Ramsey equilibrium allocation $\{s_t, k_t\}$ and tax policy $\{\tau_{t+1}\}$ ²⁷, see panel (c), both for log-utility and high risk aversion and high IES $\rho = 20$, $\sigma = 50$. For each parametrization the initial condition is the steady state capital stock absent capital income taxes. From panels (a) and (c) we observe that the optimal policy in the presence of income risk ($\sigma_{\ln \eta}^2 > 0$) is to implement a lower saving rate (through a positive capital income tax²⁸) in the initial period of the transition than would emerge in the competitive equilibrium, but also relative to the long-run optimum, i.e., $s_1^* < s_{\infty}^*$. This policy brings down the capital stock quite strongly in the initial

capital income have the same sign.

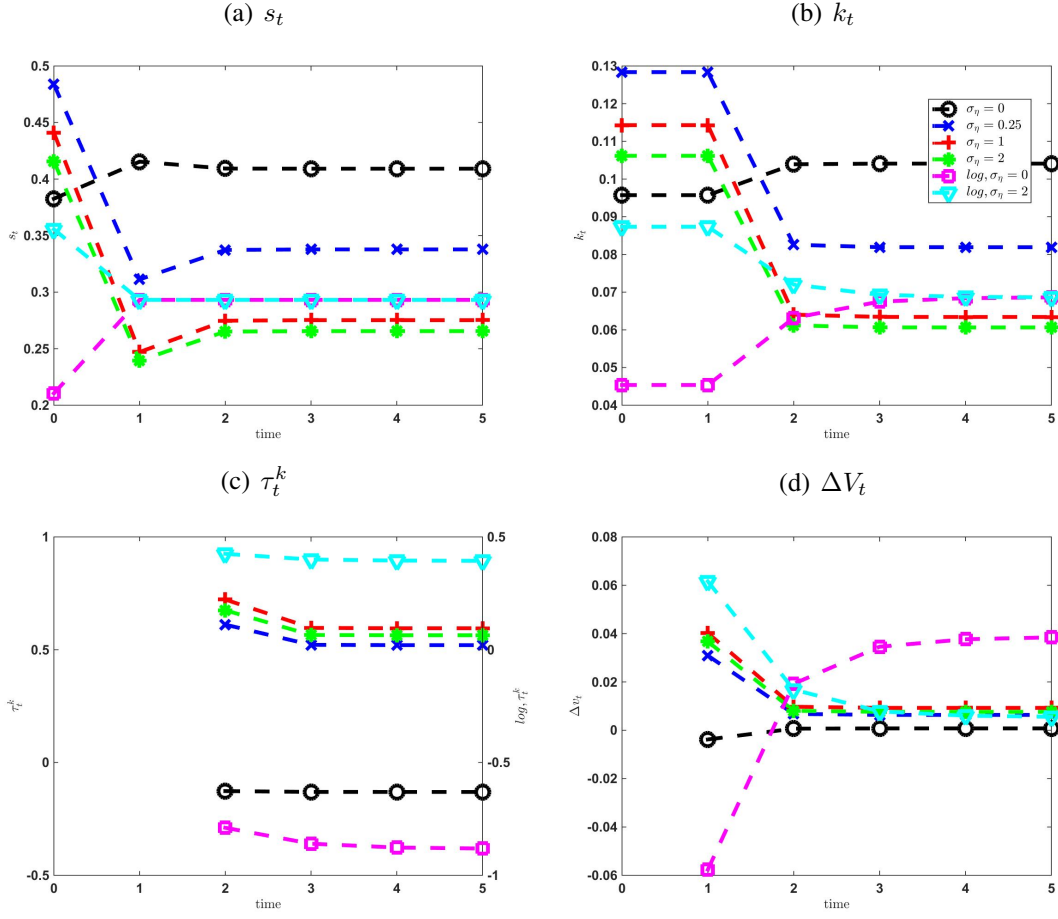
²⁵For our parameterization the threshold is at $\bar{\sigma}_{\ln \eta} \approx 0.32$.

²⁶As comparison, Table 1 also reports results for log utility for $\sigma_{\ln \eta}^2 \in \{0, 1\}$. For $\sigma_{\ln \eta}^2 = 0$ the optimal long-run saving rate exceeds the saving rate in the initial competitive equilibrium, whereas for $\sigma_{\ln \eta}^2 = 1$ it is lower, corresponding to cases 3 and 2 of Proposition 4, respectively. Consistent with Corollary 4, the long-run optimal capital income tax rate τ_{∞}^* is increasing in risk.

²⁷To compute taxes, we make use of a general implementation result, see Proposition 23 in Appendix F.4.

²⁸Left y-axis of Panel (c): $\rho = 20$, $\sigma = 50$ case. Right y-axis: Log-utility, $\rho = \sigma = 1$.

Figure 1: Policy Transition for $\rho = 20$, $\sigma = 50$ and Log Utility ($\sigma = \rho = 1$)



Notes: Initial, optimal saving rate, capital stock, optimal capital income tax, changes in lifetime utility for $\alpha = 0.2$, $\beta = 0.8$, $\kappa = 0.5$, $\sigma_{\ln \eta} \in \{0, 0.25, 1, 2\}$, $\theta = 0.9$ and $\rho = 20$, $\sigma = 50$, log-utility ($\rho = \sigma = 1$).

period (see panel (b)). Also note that for all periods t along the transition, optimal capital income taxes are inverse u-shaped in income risk, showing that the long-run results of Table 1 extend to the entire transition path. Finally, panel (d) of the figure shows the difference in lifetime utility of a generation born in period t of the optimal tax transition, relative to living in the steady state competitive equilibrium. It illustrates that for all economies with $\sigma_{\ln \eta} \geq 0.25$ the optimal Ramsey tax transition constitutes a *Pareto improvement* relative to the competitive equilibrium without taxes. Hence our analytical results on Pareto improving tax transitions from maximizing steady state utility from Section 5.2 carry over to these economies with $\theta < 1$.

These illustrative quantitative findings were derived under an arguably extreme preference parametrization ($\sigma = 50, \rho = 20$), which permitted the possibility that optimal tax rates are decreasing in the amount of income risk. Appendix H shows that the main conclusions in this section (apart from this inverse U-shape in income risk) are robust to more common values of risk aversion and of the IES (e.g. CRRA utility with $\sigma = 2$ and $\rho = 0.5$). For these parameterizations the optimal tax rate is monotonically increasing in income risk, and tends to be negative unless income risk is sufficiently large. If income risk is sufficiently large, a tax reform from the status quo of no capital taxation to the optimal Ramsey policy with positive capital income taxes again constitutes a Pareto improvement.²⁹

7 Conclusion

In this paper we have analyzed optimal capital taxes in a canonical OLG model with idiosyncratic labor income risk. We obtain a full analytical characterization of the Ramsey allocation and tax policy along the transition to a steady state when the IES is one. The optimal aggregate saving rate is independent of idiosyncratic income risk, and is implemented by a tax rate that is increasing in income risk (unless both the IES and risk aversion are large), and positive if and only if income risk is sufficiently large.

By showing that the Ramsey government can implement constrained efficient allocations through a proportional tax on capital we confirm that capital income taxation, in the context of our model, is the appropriate fiscal tool to deal with the externality on equilibrium factor prices induced by private precautionary savings behavior against uninsurable idiosyncratic income risk. However, we also demonstrate that capital should not necessarily be taxed, and should be subsidized when the government cares strongly about future generations. Judiciously chosen assumptions permit us to make these points in a fully analytically tractable and transparent manner. The next, and complementary step in this area of research would be to investigate numerically, whether in richer life cycle models with idiosyncratic income risk and thus heterogeneity in income and wealth *within generations* the optimal Ramsey tax policy is well approximated by the simple linear and time-constant tax on capital that we have shown theoretically to be optimal in our simple OLG economy.

²⁹We also consider $\sigma = 2, \rho = 20$ to show that the non-monotonicity of the tax rate with respect to income risk disappears for lower risk aversion.

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Online Appendix: Not for Publication

A Derivation of the Current Generations $CG(s)$ Effects

From equations (18b) and (18c) we find that

$$\begin{aligned}w'(s) &= (1 - \alpha)\alpha [k'(s)]^{\alpha-1} \frac{dk'(s)}{ds} = (1 - \alpha)\alpha [(1 - \kappa)(1 - \alpha)k^\alpha]^\alpha [s]^{\alpha-1} \\R'(s) &= \alpha(\alpha - 1) [k'(s)]^{\alpha-2} \frac{dk'(s)}{ds} = \alpha(\alpha - 1) [(1 - \kappa)(1 - \alpha)k^\alpha]^{\alpha-1} [s]^{\alpha-2}\end{aligned}$$

and thus

$$\kappa\eta w'(s) + (1 - \kappa)(1 - \alpha)k^\alpha R'(s)s = (1 - \alpha)\alpha [(1 - \kappa)(1 - \alpha)k^\alpha]^\alpha [s]^{\alpha-1} [\kappa\eta - 1]$$

which leads to the equation in the main text:

$$CG(s) = (1 - \alpha)\alpha [(1 - \kappa)(1 - \alpha)k^\alpha]^\alpha [s]^{\alpha-1} \beta \int u'(c^o(\eta)) [\kappa\eta - 1] d\Psi(\eta)$$

B Derivation of Optimal Saving Rate for Log-Utility

B.1 Sequential Formulation

In this section we provide a full solution to the Ramsey optimal taxation problem for the case of logarithmic utility in its sequential formulation, for an arbitrary set of social welfare weights. We first recognize from the aggregate law of motion that

$$\begin{aligned}\ln(k_{t+1}) &= \ln(1 - \alpha) + \ln(1 - \kappa) + \alpha \ln(k_t) + \ln(s_t) \\&= \varkappa + \sum_{i=0}^t \alpha^i \ln(s_{t-i}) + \alpha^{t+1} \ln(k_0) \\&= \varkappa_{t+1} + \sum_{i=0}^t \alpha^i \ln(s_{t-i})\end{aligned}$$

where $\varkappa_{t+1} = \ln(1 - \alpha) + \ln(1 - \kappa) + \alpha^{t+1} \ln(k_0)$. Therefore the objective of the Ramsey is given by (suppressing maximization-irrelevant constants)

$$\begin{aligned} \sum_{t=0}^{\infty} \omega_t V(k_t, s_t) &= \sum_{t=0}^{\infty} \omega_t [\ln(1 - s_t) + \alpha\beta \ln(s_t) + \alpha(1 + \alpha\beta) \ln(k_t)] \\ &= \chi + \sum_{t=0}^{\infty} \omega_t \left[\ln(1 - s_t) + \alpha\beta \ln(s_t) + \alpha(1 + \alpha\beta) \sum_{i=1}^{\infty} \alpha^{i-1} \ln(s_{t-i}) \right] \\ &= \chi + \sum_{t=0}^{\infty} \left[\omega_t \ln(1 - s_t) + \ln(s_t) \left(\alpha\beta\omega_t + \alpha(1 + \alpha\beta) \sum_{i=t+1}^{\infty} \omega_i \alpha^{i-(t+1)} \right) \right] \end{aligned}$$

and thus the social welfare function can be expressed purely in terms of saving rates as

$$W(\{s_t\}_{t=0}^{\infty}) = \chi + \sum_{t=0}^{\infty} \omega_t \left[\ln(1 - s_t) + \ln(s_t) \left(\alpha\beta + \alpha(1 + \alpha\beta) \sum_{j=1}^{\infty} \frac{\omega_{t+j}}{\omega_t} \alpha^{j-1} \right) \right],$$

where χ is a constant that depends positively on the initial capital stock k_0 , but is again irrelevant for maximization. Taking first order conditions with respect to s_t and setting it to zero delivers the optimal saving rate in the main text:

$$s_t = \frac{1}{1 + \left(\alpha\beta + \alpha(1 + \alpha\beta) \sum_{j=1}^{\infty} \frac{\omega_{t+j}}{\omega_t} \alpha^{j-1} \right)^{-1}}.$$

B.2 Recursive Formulation

To obtain the closed form solution of the recursive version of the problem for $\frac{\omega_{t+1}}{\omega_t} = \theta$ by the method of undetermined coefficients guess that the value function takes the following log-linear form:

$$W(k) = \Theta_0 + \Theta_1 \ln(k).$$

Using this guess and equations (18a)-(18c) rewrite the Bellman equation (17) as:

$$\begin{aligned}
W(k) &= \Theta_0 + \Theta_1 \ln(k) \\
&= \max_{s \in [0,1]} \left\{ \ln((1-s)(1-\kappa)(1-\alpha)k^\alpha) \right. \\
&\quad \left. + \beta \int \ln(\kappa\eta w(s) + R(s)s(1-\kappa)(1-\alpha)k^\alpha) d\Psi(\eta) + \theta W(k') \right\} \\
&= \ln((1-\kappa)(1-\alpha)) + \alpha\beta \ln((1-\kappa)(1-\alpha)) \\
&\quad + \int \ln(\kappa\eta(1-\alpha) + \alpha) d\Psi(\eta) + \theta\Theta_0 + \theta\Theta_1 \ln[(1-\kappa)(1-\alpha)] \\
&\quad + [\alpha + \alpha^2\beta + \alpha\theta\Theta_1] \ln(k) + \max_{s \in [0,1]} \{ \ln(1-s) + (\alpha\beta + \theta\Theta_1) \ln(s) \}.
\end{aligned} \tag{33}$$

For the Bellman equation to hold, the coefficient Θ_1 has to satisfy

$$\Theta_1 = \frac{\alpha(1 + \alpha\beta)}{(1 - \alpha\theta)}.$$

We also immediately recognize that the optimal saving rate chosen by the Ramsey planner is independent of the capital stock k and determined by the first order condition

$$\frac{1}{1-s} = \frac{\alpha\beta + \theta\Theta_1}{s}$$

and thus

$$s^* = \frac{\alpha\beta + \theta\Theta_1}{1 + \alpha\beta + \theta\Theta_1} = \frac{\alpha(\beta + \theta)}{1 + \alpha\beta}. \tag{34}$$

as given by equation (19) in the main text. Plugging in s^* and Θ_1 into the Bellman equation (33) yields a linear equation in the constant Θ_0 whose solution completes the full analytical characterization of the Ramsey optimal taxation problem.

C Dynamic Inefficiency of the Competitive Equilibrium and Positive Capital Taxation

In this section we provide the details of the relation between the solution to the Ramsey problem in the steady state and the dynamic efficiency of the steady state equilibrium absent government policy, including the proof of Proposition 4 in the main text.

First, and as usual, define the golden rule capital stock as the capital stock that maxi-

mizes aggregate (per capita) steady state consumption $C = k^\alpha - k$. Thus, the golden rule capital stock, saving rate and associated gross real interest rate are given by:

$$\begin{aligned} k^{GR} &= \alpha^{\frac{1}{1-\alpha}} \\ s^{GR} &= \frac{\alpha}{(1-\kappa)(1-\alpha)} \\ R^{GR} &= 1. \end{aligned}$$

A capital stock (respectively, a saving rate) is inefficiently high if it is larger than the golden rule level, and thus the associated gross real interest rate is less than 1. In accordance with the OLG literature we call such a capital stock, saving rate and interest rate dynamically inefficient, as aggregate consumption could be increased by lowering the capital stock in this case.

Now let us turn to the steady state of a competitive equilibrium. In any such steady state, the gross real interest rate is related to the steady state capital stock k through

$$R = \alpha k^{\alpha-1}.$$

From the law of motion of capital (equation (7)) we have

$$k = s(1-\kappa)(1-\alpha)k^\alpha$$

and thus in any steady state equilibrium the saving and interest rate are related as:

$$R = \frac{\alpha}{s(1-\kappa)(1-\alpha)}.$$

The steady state equilibrium saving rate in turn is given by (see equation (12))

$$s = \frac{1}{1 + [(1-\tau)\alpha\beta\Gamma]^{-1}} = \frac{(1-\tau)\alpha\beta\Gamma}{1 + (1-\tau)\alpha\beta\Gamma}$$

which leads to a steady state relation between the real interest rate and the tax rate determined by

$$R = \frac{\frac{1}{(1-\tau)\beta\Gamma} + \alpha}{(1-\kappa)(1-\alpha)} = R(\tau; \Gamma).$$

Consequently, a higher tax rate reduces the saving rate, the capital stock and thus increases the real interest rate. Furthermore, for a given τ , the steady state interest rate is decreasing

in the amount of income risk (unless $\beta = 0$).

The steady state interest rate in the absence of government policy is given by

$$R(\tau = 0; \Gamma) = \frac{\frac{1}{\beta\Gamma} + \alpha}{(1 - \kappa)(1 - \alpha)}$$

and thus the steady state competitive equilibrium without taxes is dynamically inefficient ($R(\tau = 0; \Gamma) < 1$) if and only if

$$\frac{\frac{1}{\beta\Gamma} + \alpha}{(1 - \kappa)(1 - \alpha)} < 1$$

or if and only if

$$\begin{aligned} \frac{1}{[(1 - \kappa)(1 - \alpha) - \alpha]\Gamma} &< \beta \\ \Theta_1(\Gamma) &:= \frac{1}{(1 - \alpha)\Gamma - \Gamma/\bar{\Gamma}} < \beta \end{aligned} \quad (35)$$

where $\bar{\Gamma} \leq \Gamma$, with equality if η is degenerate at $\eta = 1$, and thus there is no income risk.

The optimal Ramsey steady state (i.e., $\theta = 1$) saving and tax rates (see equations (19) and (20)) are given by

$$\begin{aligned} s^* &= \frac{\alpha(1 + \beta)}{1 + \alpha\beta} \\ 1 - \tau &= \frac{1 + \beta}{(1 - \alpha)\beta\Gamma} \end{aligned}$$

and thus the optimal Ramsey tax rate is positive, $\tau > 0$, if and only if

$$\frac{(1 + \beta)}{(1 - \alpha)\beta\Gamma} < 1$$

or if and only if

$$\Theta_2 := \frac{1}{(1 - \alpha)\Gamma - 1} < \beta. \quad (36)$$

Since

$$\Theta_2(\Gamma) = \frac{1}{(1 - \alpha)\Gamma - 1} \leq \frac{1}{(1 - \alpha)\Gamma - \Gamma/\bar{\Gamma}} = \Theta_1(\Gamma)$$

with equality if and only if η is degenerate at $\eta = 1$, we conclude that if the competitive

equilibrium absent taxes is dynamically inefficient, the optimal Ramsey steady state capital tax rate is positive. The reverse is not true, however: with income risk the optimal Ramsey tax rate might be positive even when the steady state equilibrium absent policy is dynamically efficient. To see this, comparing saving rates we have

$$\begin{aligned} s^* &= \frac{\alpha(1 + \beta)}{1 + \alpha\beta} \\ s^{CE} &= \frac{1}{1 + [\alpha\beta\Gamma]^{-1}} \\ s^{GR} &= \frac{\alpha}{(1 - \kappa)(1 - \alpha)} \end{aligned}$$

and thus $s^{CE} > s^{GR}$ if and only if

$$\beta > \frac{1}{[(1 - \kappa)(1 - \alpha) - \alpha]\Gamma}$$

and thus if and only if the steady state equilibrium is dynamically inefficient. Furthermore $s^* < s^{CE}$ if and only if $\Theta_2(\Gamma) < \beta$ and thus if and only if $\tau > 0$.

Stating inequalities (35) and (36) in terms of Γ gives Proposition 4 in the main text. Furthermore, we collect the relationship between dynamic inefficiency and a positive Ramsey steady state capital tax rate in the following

Proposition 12. *Let $\theta = 1$. If the steady state competitive equilibrium is dynamically inefficient, then the optimal Ramsey tax rate τ is positive. If in addition η is degenerate at $\eta = 1$, then the reverse is true as well: $\tau > 0$ only if the steady state competitive equilibrium is dynamically inefficient.*

D Robustness

D.1 Ex-Ante Heterogeneity

Permanent productivity is denoted by ν and we assume that the cdf of ν is given by $\Phi(\nu)$. We assume that a LLN applies so that Φ is both the population distribution of permanent productivity ν as well as the ex-ante cdf over ν for each household. We make the following

Assumption 5. *The shock ν takes positive values Φ -almost surely and*

$$\int \nu d\Phi = 1.$$

Furthermore, shocks η and ν are independent, thus

$$\int_{\nu} \int_{\eta} \nu \eta d\Phi(\nu) d\Psi(\eta) = \int \nu d\Phi(\nu) \cdot \int \eta d\Psi(\eta) = 1.$$

The budget constraints of each agent of productivity type i is now given by

$$\begin{aligned} a_{t+1}(\nu) + c_t^y(\nu) &= (1 - \kappa)\nu w_t \\ c_{t+1}^o(\nu, \eta) &= a_{t+1}R_{t+1}(1 - \tau_{t+1}) + \eta_{t+1}\nu\kappa w_{t+1} + T_{t+1}(\nu), \end{aligned}$$

where

$$T_{t+1}(\nu) = a_{t+1}(\nu)R_{t+1}\tau_{t+1}$$

In all periods t we have $L_t = \int \int ((1 - \kappa)\nu + \kappa\nu\eta_t) d\Psi(\eta) d\Phi(\nu) = 1$ and thus the capital stock in period $t + 1$, K_{t+1} is equal to the capital intensity $k_{t+1} = \frac{K_{t+1}}{L_{t+1}}$. Denote by

$$s_t(\nu) = \frac{a_{t+1}(\nu)}{(1 - \kappa)\nu w_t}$$

the saving rate of household of type ν . The capital intensity in period $t + 1$ is then

$$k_{t+1} = \int a_{t+1}(\nu) d\Phi(\nu) = (1 - \kappa)(1 - \alpha)k_t^\alpha \int s_t(\nu)\nu d\Phi(\nu).$$

D.1.1 General Equilibrium

Proposition 13. *The general equilibrium saving rates $s_t(\nu)$ are identical for all agents: $s_t(\nu) = s_t$ for all ν .*

Proof. If $s(\nu) = s_t$ then since $\int \nu d\Phi(\nu) = 1$ the law of motion of the capital stock is

$$k_{t+1} = s_t(1 - \kappa)(1 - \alpha)k_t^\alpha.$$

The first-order condition with log utility of each household is now

$$\begin{aligned}
1 &= \beta R_{t+1}(1 - \tau_{t+1}) \int \int \frac{c_t^y(\nu)}{c_{t+1}^o(\eta, \nu)} d\Psi(\eta) d\Phi(\nu) \\
&= \alpha \beta k_{t+1}^{\alpha-1} (1 - \tau_{t+1}) \int \int \frac{(1 - s_t) \nu (1 - \kappa) (1 - \alpha) k_t^\alpha}{s_t (1 - \kappa) (1 - \alpha) k_t^\alpha \alpha k_{t+1}^{\alpha-1} + \eta_{t+1} \kappa (1 - \alpha) k_{t+1}^\alpha} d\Psi(\eta) d\Phi(\nu) \\
&= \alpha \beta k_{t+1}^{\alpha-1} (1 - \tau_{t+1}) \int \frac{\frac{(1-s_t)}{s_t} k_{t+1}}{k_{t+1} \alpha k_{t+1}^{\alpha-1} + \eta_{t+1} \kappa (1 - \alpha) k_{t+1}^\alpha} d\Psi(\eta) \\
&= \alpha \beta (1 - \tau_{t+1}) \frac{1 - s_t}{s_t} \Gamma.
\end{aligned}$$

Thus the optimal saving rate is independent of permanent productivity ν . \square

D.1.2 Ramsey Problem

Proposition 14. *Permanent ex-ante heterogeneity in productivity ν does not affect the optimal choice of s .*

Proof. The objective of the Ramsey planner is now given by

$$\begin{aligned}
W(k) &= \max_{s \in (0,1)} \int \ln((1-s)\nu(1-\kappa)(1-\alpha)k^\alpha) d\Phi(\nu) + \\
&\quad \beta \int \int \ln(\kappa \eta \nu w(s) + R(s)s\nu(1-\kappa)(1-\alpha)k^\alpha) d\Phi(\nu) d\Psi(\eta), \\
&= (1 + \beta) \int \ln(\nu) d\Phi(\nu) + \max_{s \in (0,1)} \int \ln((1-s)(1-\kappa)(1-\alpha)k^\alpha) + \\
&\quad \beta \int \ln(\kappa \eta w(s) + R(s)s(1-\kappa)(1-\alpha)k^\alpha) d\Psi(\eta)
\end{aligned}$$

and thus heterogeneity with respect to ν does not affect the optimization. \square

D.2 Idiosyncratic Return Risk

We denote return shocks by ϱ_{t+1} and assume that they are iid. We assume that the cdf of ϱ is given by $\Upsilon(\varrho)$ and denote the corresponding pdf by $v(\varrho)$. We again assume that a LLN applies so that Υ is both the population distribution of ϱ as well as the individual cdf of return shocks. We make the following

Assumption 6. *The shock ϱ takes positive values Υ -almost surely and*

$$\int \varrho d\Upsilon = 1.$$

Furthermore, shocks η and ϱ are independent³⁰ and therefore

$$\int_{\varrho} \int_{\eta} \varrho \eta d\Upsilon(\varrho) d\Psi(\eta) = \int \varrho d\Upsilon(\varrho) \cdot \int \eta d\Psi(\eta)$$

almost surely.

The budget constraints now write as

$$\begin{aligned} a_{t+1} + c_t^y &= (1 - \kappa)w_t \\ c_{t+1}^o(\eta, \varrho) &= a_{t+1}R_{t+1}\varrho_{t+1}(1 - \tau_{t+1}) + \eta_{t+1}\kappa w_{t+1} + T_{t+1}(\varrho) \end{aligned}$$

and we assume that transfer payments are contingent on the rate of return realization,

$$T_{t+1}(\varrho) = a_{t+1}R_{t+1}\varrho_{t+1}\tau_{t+1}.$$

D.2.1 General Equilibrium

Proposition 15. *The structure of the competitive equilibrium is unchanged, but now idiosyncratic risk summarized by Γ is expressed in terms of the distribution $\Pi(\delta_{t+1})$ of the random variable $\delta_{t+1} = \frac{\eta_{t+1}}{\varrho_{t+1}}$ instead of $\Psi(\eta_{t+1})$.*

Proof. The first-order condition for log utility is now

$$\begin{aligned} 1 &= \beta R_{t+1}(1 - \tau_{t+1}) \int \int \varrho_{t+1} \frac{c_t^y}{c_{t+1}^o(\eta)} d\Psi(\eta) d\Upsilon(\varrho) \\ &= \alpha \beta k_{t+1}^{\alpha-1} (1 - \tau_{t+1}) \int \int \varrho_{t+1} \frac{(1 - s_t)(1 - \kappa)(1 - \alpha)k_t^\alpha}{s_t(1 - \kappa)(1 - \alpha)k_t^\alpha \alpha k_{t+1}^{\alpha-1} \varrho_{t+1} + \eta_{t+1}\kappa(1 - \alpha)k_{t+1}^\alpha} d\Psi(\eta) d\Upsilon(\varrho) \\ &= \alpha \beta (1 - \tau_{t+1}) \frac{1 - s_t}{s_t} \int \int \left(\alpha + \kappa(1 - \alpha) \frac{\eta_{t+1}}{\varrho_{t+1}} \right)^{-1} d\Psi(\eta) d\Upsilon(\varrho) \\ &= \alpha \beta (1 - \tau_{t+1}) \frac{1 - s_t}{s_t} \int (\alpha + \kappa(1 - \alpha) \delta_{t+1})^{-1} d\Pi(\delta) \\ &= \alpha \beta (1 - \tau_{t+1}) \frac{1 - s_t}{s_t} \Gamma(\alpha, \kappa; \delta, \Pi). \end{aligned}$$

³⁰Independence is assumed for simplicity of notation but can be relaxed for the result.

and thus the general equilibrium saving rate is the same as before, with Γ expressed in terms of random variable δ and its cdf $\Pi(\delta)$. \square

D.2.2 Ramsey Problem

Proposition 16. *The structure of the optimal Ramsey problem is unchanged. Again the stochasticity is expressed in terms of the random variable $\delta_{t+1} = \frac{\eta_{t+1}}{\varrho_{t+1}}$ instead of η_{t+1} .*

Proof. The objective of the Ramsey planner is given by

$$\begin{aligned}
W(k) &= \max_{s \in (0,1)} \ln((1-s)(1-\kappa)(1-\alpha)k^\alpha) + \\
&\quad \beta \int \int \ln(\kappa \eta w(s) + R(s) \varrho s (1-\kappa)(1-\alpha)k^\alpha) d\Upsilon(\varrho) d\Psi(\eta) \\
&= \max_{s \in (0,1)} \ln((1-s)(1-\kappa)(1-\alpha)k^\alpha) + \\
&\quad \beta \int \int \ln\left(\varrho \left(\kappa \frac{\eta}{\varrho} w(s) + R(s) s (1-\kappa)(1-\alpha)k^\alpha\right)\right) d\Upsilon(\varrho) d\Psi(\eta) \\
&= \beta \int \ln(\varrho) d\Upsilon(\varrho) + \max_{s \in (0,1)} \ln((1-s)(1-\kappa)(1-\alpha)k^\alpha) + \\
&\quad \beta \int \ln(\kappa \delta w(s) + R(s) s (1-\kappa)(1-\alpha)k^\alpha) d\Pi(\delta)
\end{aligned}$$

\square

D.3 Time Varying Technological Progress and Population Growth

Denote by A_t the level of technology (labor productivity) and assume that it evolves deterministically according to $A_t = (1 + g_t)A_{t-1}$, where the growth rate of technology g_t is allowed to be time-varying. The population growth rate $n \geq 0$ is assumed to be constant over time, so that the size of the young population evolves according to $N_t^y = (1 + n)N_{t-1}^y$. With these modifications, aggregate production is

$$Y_t = F(K_t, A_t L_t) = K_t^\alpha (A_t L_t)^{1-\alpha},$$

where L_t is aggregate labor supply given by

$$L_t = (1 - \kappa)N_t^y + \kappa N_t^o = ((1 - \kappa)(1 + n) + \kappa) N_{t-1}^y.$$

Define the capital intensity in terms of efficiency units of labor as $k_t = \frac{K_t}{A_t L_t}$. Then, under the maintained assumption of Cobb-Douglas production, $Y_t = K_t^\alpha (A_t L_t)^{1-\alpha}$ we get $y_t = \frac{Y_t}{A_t L_t} = k_t^\alpha$ and thus wages (per effective unit of labor) and interest rates are

$$\begin{aligned} w_t &= (1 - \alpha) k_t^\alpha A_t \\ R_t &= \alpha k_t^{\alpha-1}. \end{aligned}$$

The law of motion of the capital intensity can be derived as

$$\begin{aligned} K_{t+1} &= a_{t+1} N_t^y = s_t (1 - \kappa) (1 - \alpha) k_t^\alpha A_t N_t^y \\ \Leftrightarrow \quad k_{t+1} &= s_t \frac{(1 - \kappa)(1 - \alpha)}{(1 + g_{t+1})((1 - \kappa)(1 + n) + \kappa)} k_t^\alpha. \end{aligned}$$

D.3.1 General Equilibrium

Proposition 17. *A time varying rate of technological progress g_t does not affect the saving rate in the competitive general equilibrium, whereas an increase of the constant population growth rate n increases the saving rate.*

Proof. Start from the FOC, equation (5), given by

$$1 = \beta(1 - \tau_{t+1}) \int \frac{1 - s_t}{s_t(1 - \tau_{t+1}) + \frac{\kappa w_{t+1}}{(1 - \kappa)w_t R_{t+1}} \eta_{t+1} + \frac{T_{t+1}}{(1 - \kappa)w_t R_{t+1}}} d\Psi(\eta_{t+1})$$

and use that

$$\tau_{t+1} s_t = \frac{T_{t+1}}{(1 - \kappa)w_t R_{t+1}}$$

to obtain

$$1 = \beta(1 - \tau_{t+1}) \int \frac{1 - s_t}{s_t + \frac{\kappa w_{t+1}}{(1 - \kappa)w_t R_{t+1}} \eta_{t+1}} d\Psi(\eta_{t+1})$$

Next, rewrite $\frac{w_{t+1}}{w_t R_{t+1}}$ as

$$\begin{aligned}\frac{w_{t+1}}{w_t R_{t+1}} &= \frac{k_{t+1}^\alpha A_{t+1}}{k_t^\alpha A_t \alpha k_{t+1}^{\alpha-1}} = (1 + g_{t+1}) \frac{1}{\alpha} \frac{k_{t+1}}{k_t^\alpha} \\ &= (1 + g_{t+1}) \frac{1}{\alpha} s_t (1 - \kappa) (1 - \alpha) \frac{1}{(1 + g_{t+1}) ((1 - \kappa)(1 + n) + \kappa)} \\ &= \frac{1}{\alpha} s_t (1 - \kappa) (1 - \alpha) \frac{1}{(1 - \kappa)(1 + n) + \kappa}.\end{aligned}$$

Observe that the time varying growth rate g_{t+1} cancels out, and we can rewrite the FOC as

$$\begin{aligned}1 &= \alpha \beta (1 - \tau_{t+1}) \frac{1 - s_t}{s_t} \int \frac{1}{\alpha + \kappa (1 - \alpha) \frac{1}{(1 - \kappa)(1 + n) + \kappa} \eta_{t+1}} d\Psi(\eta_{t+1}) \\ &= \alpha \beta (1 - \tau_{t+1}) \frac{1 - s_t}{s_t} \check{\Gamma}.\end{aligned}$$

where $\check{\Gamma} := \int \frac{1}{\alpha + \kappa (1 - \alpha) \frac{1}{(1 - \kappa)(1 + n) + \kappa} \eta_{t+1}} d\Psi(\eta_{t+1})$. □

D.3.2 Ramsey Optimum

Proposition 18. *A time varying rate of technological progress g_t as well as a constant population growth rate n leave the optimal Ramsey saving rate unchanged.*

Proof. With log utility, cohort t lifetime utility is given by

$$\begin{aligned}V_t(k_t, s_t, A_t) &= \ln(A_t) + \ln((1 - s_t)(1 - \kappa)k_t^\alpha) + \alpha \beta \ln((1 + g_{t+1})k_{t+1}(s_t)) + \beta \ln(\Gamma_2) \\ &= \ln(A_t) + \alpha \beta \ln(1 + g_{t+1}) + \tilde{V}_t(k_t, s_t),\end{aligned}$$

where $\Gamma_2 = \int ((1 - \alpha)\kappa\eta_{t+1} + \alpha)^{1-\sigma} d\Psi(\eta_{t+1})$. Next, assume that the government maximizes the discounted sum of utility of cohorts t weighted by the population size of that cohort so that the objective is to maximize

$$W_0 = \sum_{t=0}^{\infty} \omega_t N_t^y V_t(k_t, s_t, A_t) = \chi + \sum_{t=0}^{\infty} \omega_t N_t^y \tilde{V}_t(k_t, s_t)$$

where χ is a maximization irrelevant constant. Finally, normalizing $N_0 = 1$ we get

$$W_0 = \sum_{t=0}^{\infty} \tilde{\omega}_t \tilde{V}_t(k_t, s_t)$$

where $\tilde{\omega}_t = \omega_t(1+n)^t$. Also note that

$$k_{t+1}(s_t) = s_t \frac{(1-\kappa)(1-\alpha)}{(1+g_{t+1})((1-\kappa)(1+n)+\kappa)} k_t^\alpha.$$

and thus

$$\begin{aligned} \tilde{V}_t(k_t, s_t) &= \ln(((1-s_t)(1-\kappa)k_t^\alpha)) + \alpha\beta \ln(k_{t+1}(s_t)) + \beta \ln(\Gamma_2) \\ &= \ln(((1-s_t)(1-\kappa)k_t^\alpha)) + \alpha\beta \ln\left(s_t \frac{(1-\alpha)(1-\kappa)}{(1+g_{t+1})((1-\kappa)(1+n)+\kappa)} k_t^\alpha\right) + \beta \ln(\Gamma_2) \\ &= \chi_t + \ln(1-s_t) + \alpha(1+\alpha\beta) \ln(k_t) + \alpha\beta \ln(s_t) \end{aligned}$$

and thus time varying technological progress and population growth only add a maximization irrelevant (time varying) additive parameter. Also since

$$\begin{aligned} \ln(k_{t+1}) &= \ln(1-\alpha) + \ln(1-\kappa) + \alpha \ln(k_t) + \ln(s_t) - \ln((1+g_{t+1})((1-\kappa)(1+n)+\kappa)) \\ &= \varkappa_{t+1} + \sum_{\tau=0}^t \alpha^\tau \ln(s_{t-\tau}) + \alpha^{t+1} \ln(k_0) \\ &= \tilde{\varkappa}_{t+1} + \sum_{\tau=0}^t \alpha^\tau \ln(s_{t-\tau}) \end{aligned}$$

we can substitute out $\ln(k_t)$ in the cohort t utility function (as before), which adds additional maximization irrelevant time varying terms. \square

D.3.3 The Bounds of Proposition 4 with Technological Progress and Population Growth

We focus on a steady state where the rate of technological progress is a constant g .

Golden Rule. Maximizing steady state utility is equivalent to maximizing per capita consumption. The per capita resource constraint, noticing that in the social planner's optimum $c_t^o(\eta) = c_t^o$, is

$$\frac{c_t^y N_t^y + c_t^o N_t^o}{N_t} = \frac{F(K_t, L_t) - K_{t+1}}{N_t}.$$

Now observe that in steady state where $k_{t+1} = k_t = k$ we have

$$\begin{aligned}
N_t^y &= (1+n)N_{t-1}^y, \quad N_t^o = N_{t-1}^y \\
N_t &= N_t^y + N_t^o = (2+n)N_{t-1}^y \\
L_t &= ((1-\kappa)(1+n) + \kappa) N_{t-1}^y \\
F(K_t, L_t) &= k^\alpha A_t L_t = k^\alpha A_t ((1-\kappa)(1+n) + \kappa) N_{t-1}^y \\
K_{t+1} &= k A_{t+1} L_{t+1} = k(1+n)(1+g) ((1-\kappa)(1+n) + \kappa) N_{t-1}^y
\end{aligned}$$

and thus maximizing per capita consumption is equivalent to

$$\max_k \{ \tilde{c}_t^y(1+n) + \tilde{c}_t^o \} = \max_k \{ (k^\alpha - k(1+n)(1+g)) ((1-\kappa)(1+n) + \kappa) \}$$

where $\tilde{c}_t = \frac{c_t}{A_t}$ is de-trended consumption. The first-order condition gives

$$\alpha k^{\alpha-1} = (1+n)(1+g)$$

and thus the golden-rule capital stock is

$$k^{GR} = \left(\frac{\alpha}{(1+n)(1+g)} \right)^{\frac{1}{1-\alpha}}$$

with the standard intuitive explanation that, with population growth and technological progress, more efficient workers have to be equipped each period with an increasing capital stock to hold constant capital per efficient worker. The golden rule interest rate is thus

$$R^{GR} = \alpha k^{GR\alpha-1} = (1+n)(1+g).$$

Finally, from the law of motion of the capital stock we have

$$k' = s \frac{(1-\kappa)(1-\alpha)}{(1+g)((1-\kappa)(1+n) + \kappa)} k^\alpha$$

and thus the steady state capital stock for given saving rate is

$$k^* = \left(s^* \frac{(1-\kappa)(1-\alpha)}{(1+g)((1-\kappa)(1+n) + \kappa)} \right)^{\frac{1}{1-\alpha}}.$$

Setting $k^* = k^{GR}$ then gives the golden rule saving rate as

$$s^{GR} = \frac{\alpha((1-\kappa)(1+n) + \kappa)}{(1-\kappa)(1-\alpha)(1+n)}.$$

Competitive Equilibrium and Dynamic Efficiency. Since $R^* = \alpha k^{*\alpha-1}$, the steady state interest rate for given saving rate is

$$R^* = \frac{\alpha(1+g)((1-\kappa)(1+n) + \kappa)}{s^*(1-\kappa)(1-\alpha)}.$$

Now use that

$$s^* = \frac{(1-\tau)\alpha\beta\check{\Gamma}}{1 + (1-\tau)\alpha\beta\check{\Gamma}},$$

as defined above, to get

$$R^*(\tau, \check{\Gamma}) = \frac{(1+g)((1-\kappa)(1+n) + \kappa)}{(1-\kappa)(1-\alpha)} \left(\alpha + \frac{1}{(1-\tau)\beta\check{\Gamma}} \right)$$

and thus in the laissez-faire steady state we have

$$R^*(\tau = 0, \check{\Gamma}) = \frac{(1+g)((1-\kappa)(1+n) + \kappa)}{(1-\kappa)(1-\alpha)} \left(\alpha + \frac{1}{\beta\check{\Gamma}} \right).$$

Since the laissez-faire equilibrium economy is dynamically inefficient if $R^*(\tau = 0, \check{\Gamma}) < (1+n)(1+g)$ we get that the economy is dynamically inefficient if

$$\begin{aligned} & \frac{(1+g)((1-\kappa)(1+n) + \kappa)}{(1-\kappa)(1-\alpha)} \left(\alpha + \frac{1}{\beta\check{\Gamma}} \right) < (1+n)(1+g) \\ \Leftrightarrow & \beta > \frac{1}{\left(\frac{(1-\kappa)(1-\alpha)(1+n)}{(1-\kappa)(1+n)+\kappa} - \alpha \right) \check{\Gamma}} \end{aligned}$$

Recall that

$$\check{\Gamma} = \int \frac{1}{\alpha + \kappa(1-\alpha) \frac{1}{(1-\kappa)(1+n)+\kappa} \eta_{t+1}} d\Psi(\eta_{t+1}).$$

and thus in the deterministic economy we have

$$\bar{\Gamma} = \frac{1}{\alpha + \kappa(1 - \alpha) \frac{1}{(1 - \kappa)(1 + n) + \kappa}}$$

Now rewrite the bound on β above to get

$$\begin{aligned} \beta &> \frac{1}{\left(\frac{(1 - \kappa)(1 - \alpha)(1 + n)}{(1 - \kappa)(1 + n) + \kappa} - \alpha \right) \check{\Gamma}} \\ &= \frac{1}{\left(\frac{-\kappa(1 - \alpha) + (1 - \alpha)(1 + n) - \kappa(1 - \alpha)n}{(1 - \kappa)(1 + n) + \kappa} - \alpha \right) \check{\Gamma}} \\ &= \frac{1}{\left(\frac{(1 - \alpha)(1 + n(1 - \kappa))}{1 + n(1 - \kappa)} - \left(\alpha + \frac{\kappa(1 - \alpha)}{(1 - \kappa)(1 + n) + \kappa} \right) \right) \check{\Gamma}} \\ &= \frac{1}{\left((1 - \alpha)\check{\Gamma} - \check{\Gamma}/\bar{\Gamma} \right)} := \Theta_1(\check{\Gamma}, \bar{\Gamma}) \end{aligned}$$

Since the structure of the Ramsey problem has not changed, we continue to find that the optimal saving rate for $\theta = 1$ is

$$s^* = \frac{\alpha(1 + \beta)}{1 + \alpha\beta}$$

and thus the tax rate implementing it satisfies

$$1 - \tau = \frac{1 + \beta}{(1 - \alpha)\beta\check{\Gamma}}$$

and thus we have $\tau > 0$, if and only if

$$\frac{1 + \beta}{(1 - \alpha)\beta\check{\Gamma}} < 1$$

or if and only if

$$\Theta_2(\check{\Gamma}) := \frac{1}{(1 - \alpha)\check{\Gamma} - 1} < \beta.$$

Stating the inequalities in terms of $\check{\Gamma}$ the regions corresponding to Proposition 4 become

1. $\check{\Gamma} > \frac{1}{((1 - \alpha) - 1/\bar{\Gamma})\beta}$: dynamic inefficiency, $\tau > 0$

$$2. \tilde{\Gamma} \in \left(\frac{1+\beta}{(1-\alpha)\beta}, \frac{1}{((1-\alpha)-1/\tilde{\Gamma})\beta} \right]: \text{dynamic efficiency, } \tau > 0$$

$$3. \tilde{\Gamma} \in \left(\bar{\tilde{\Gamma}}, \frac{1+\beta}{(1-\alpha)\beta} \right]: \text{dynamic efficiency, } \tau < 0$$

Recall that

$$\bar{\tilde{\Gamma}} = \frac{1}{\alpha + \kappa(1 - \alpha) \frac{1}{(1-\kappa)(1+n)+\kappa}}$$

and thus an increase of n increases $\bar{\tilde{\Gamma}}$ increasing the lower bound of the third interval. By increasing $\tilde{\Gamma}$ it also reduces $\frac{1}{((1-\alpha)-1/\tilde{\Gamma})\beta}$ and thus the interesting interval (the case 2 of intermediate risk) gets smaller. Finally, positive population growth reduces the sensitivity of $\tilde{\Gamma}$ with respect to increasing risk.

E Characterization of Efficient Allocations

E.1 Characterization of Pareto Efficient Allocations

In this section we derive the solution to the unconstrained social planner problem and study whether the Ramsey government implements Pareto efficient allocations. The obvious answer is no, since an unconstrained social planner would provide full insurance against idiosyncratic η shocks, which, given the market structure, is ruled out in any competitive equilibrium. More interesting is the question how the saving rate chosen by the unconstrained planner compares to that selected by a constrained planner and the Ramsey government. We again assume logarithmic utility. The planner maximizes social welfare

$$\omega_{-1} \int \ln(c_0^o(\eta_0)) d\Psi(\eta_0) + \sum_{t=0}^{\infty} \omega_t \left[\ln(c_t^y) + \beta \int \ln(c_{t+1}^o(\eta_{t+1})) d\Psi(\eta_{t+1}) \right]$$

subject just to the sequence of resource constraints

$$c_t^y + \int c_t^o(\eta_t) d\Psi(\eta_t) + k_{t+1} = k_t^\alpha.$$

We again restrict attention to geometrically declining welfare weights: $\omega_{t+1}/\omega_t = \theta \leq 1$. Trivially, the social planner provides full insurance against idiosyncratic income risk so

that $c_t^o(\eta) = c_t^o$ for all η and all t . Thus the problem simplifies to

$$\begin{aligned} \max_{\{c_t^y, c_t^o, k_{t+1}\}} \quad & \omega_{-1} \ln(c_0^o) + \sum_{t=0}^{\infty} \omega_t [\ln(c_t^y) + \beta \ln(c_{t+1}^o)] \quad s.t. \\ & c_t^y + c_t^o + k_{t+1} = k_t^\alpha \end{aligned}$$

with $k_0 > 0$ given. The first order conditions are given by

$$\begin{aligned} \frac{\omega_t}{c_t^y} &= \lambda_t \\ \frac{\beta \omega_{t-1}}{c_t^o} &= \lambda_t \\ \lambda_t &= \lambda_{t+1} \alpha k_{t+1}^{\alpha-1} \\ c_t^y + c_t^o + k_{t+1} &= k_t^\alpha. \end{aligned}$$

The optimal allocation of consumption across two generations at time t is then given by

$$\frac{c_t^o}{c_t^y} = \frac{\beta \omega_{t-1}}{\omega_t}$$

and across time it is characterized by

$$\frac{c_{t+1}^o}{c_t^y} = \beta \alpha k_{t+1}^{\alpha-1}.$$

In contrast to the Ramsey problem, consumption of the old in the first period is no longer irrelevant for maximization because the social planner can redistribute resources inter-generationally whereas the Ramsey planner, given the assumed restriction on instruments cannot. Thus, we characterize optimal allocations in period 0 and in $t > 0$ separately.

Periods $t > 0$. In all periods $t > 0$ again assume that $\frac{\omega_{t+1}}{\omega_t} = \theta$. We then get

$$\frac{c_t^o}{c_t^y} = \frac{\beta \omega_{t-1}}{\omega_t} = \frac{\beta}{\theta}$$

and thus from the resource constraint we get

$$\begin{aligned} c_t^y &= \frac{\theta}{\theta + \beta} (k_t^\alpha - k_{t+1}) \\ c_t^o &= \frac{\beta}{\theta + \beta} (k_t^\alpha - k_{t+1}). \end{aligned}$$

Define, similarly to the Ramsey problem, the saving rate of the social planner as

$$s_t = \frac{k_{t+1}}{(1 - \kappa)(1 - \alpha)k_t^\alpha}.$$

Then from the first order conditions we get

$$\begin{aligned} \frac{1}{c_t^y} &= \frac{\beta}{c_{t+1}^o} \alpha k_{t+1}^{\alpha-1} \\ \frac{k_{t+1}}{(k_t^\alpha - k_{t+1})} &= \frac{\alpha \theta k_{t+1}^\alpha}{(k_{t+1}^\alpha - k_{t+2})} \\ (1 - (1 - \kappa)(1 - \alpha)s_{t+1}) &= \alpha \theta \left(\frac{1}{(1 - \kappa)(1 - \alpha)s_t} - 1 \right). \end{aligned}$$

As in the neoclassical growth model we can show that the only solution to the first order difference equation that does not eventually violate the non-negativity constraint of consumption and does not violate the TVC is the constant saving rate s solving

$$(1 - (1 - \kappa)(1 - \alpha)s) = \alpha \theta \left(\frac{1}{(1 - \kappa)(1 - \alpha)s} - 1 \right).$$

Define $\tilde{s} = (1 - \kappa)(1 - \alpha)s$ then we have

$$1 - \tilde{s} = \alpha \theta \left(\frac{1}{\tilde{s}} - 1 \right)$$

with solutions $\tilde{s} = 1$ and $\tilde{s} = \alpha \theta$ and thus

$$s^{SP} = \frac{\alpha \theta}{(1 - \kappa)(1 - \alpha)}.$$

The optimal sequence of capital stocks, starting from k_0 , is therefore given by

$$\begin{aligned} k_{t+1} &= (1 - \kappa)(1 - \alpha)s_t k_t^\alpha \\ &= \alpha \theta k_t^\alpha. \end{aligned}$$

Since

$$k_t^\alpha - k_{t+1} = (1 - \alpha \theta) k_t^\alpha$$

we immediately have that the social planner's problem is given in all periods $t > 0$ by a constant saving rate

$$s^{SP} = \frac{k_{t+1}}{(1 - \kappa)(1 - \alpha)k_t^\alpha} = \frac{\alpha \theta}{(1 - \kappa)(1 - \alpha)}$$

and associated sequence of capital stocks

$$k_{t+1} = \alpha \theta k_t^\alpha.$$

Period 0. Let us next characterize the allocation in period $t = 0$. We get

$$\frac{c_0^o}{c_0^y} = \frac{\beta \omega_{-1}}{\omega_0}$$

and can, without loss of generality, normalize $\omega_0 = 1$ so that

$$\frac{c_0^o}{c_0^y} = \beta \omega_{-1}.$$

Now use this in the resource constraint to get

$$c_0^y = \frac{1}{1 + \beta \omega_{-1}} (k_0^\alpha - k_1)$$

$$c_0^o = \frac{\beta \omega_{-1}}{1 + \beta \omega_{-1}} (k_0^\alpha - k_1)$$

$$k_1 = s_0(1 - \kappa)(1 - \alpha)k_0^\alpha.$$

Then from the first order conditions we get

$$\begin{aligned}\frac{1}{c_0^y} &= \frac{\beta}{c_1^o} \alpha k_1^{\alpha-1} \\ \frac{k_1 (1 + \beta \omega_{-1})}{(k_0^\alpha - k_1)} &= \frac{\alpha (\theta + \beta) k_1^\alpha}{(k_1^\alpha - k_2)} \\ \frac{s_0 (1 - \kappa) (1 - \alpha) (1 + \beta \omega_{-1})}{(1 - s_0 (1 - \kappa) (1 - \alpha))} &= \frac{\alpha (\theta + \beta)}{(1 - \alpha \theta)}\end{aligned}$$

and thus

$$s_0 = \frac{\alpha (\theta + \beta)}{(1 - \alpha \theta)} \left[(1 - \kappa) (1 - \alpha) \left((1 + \beta \omega_{-1}) + \frac{\alpha (\theta + \beta)}{(1 - \alpha \theta)} \right) \right]^{-1}.$$

We summarize these results in the following

Proposition 19. *The solution to the social planner problem, for any $k_0 > 0$, is given by*

$$s_0^{SP} = \frac{\alpha (\theta + \beta)}{(1 - \alpha \theta)} \left[(1 - \kappa) (1 - \alpha) \left((1 + \beta \omega_{-1}) + \frac{\alpha (\theta + \beta)}{(1 - \alpha \theta)} \right) \right]^{-1}$$

and associated capital stock in period 1

$$k_1 = s_0^{SP} (1 - \kappa) (1 - \alpha) k_0^\alpha.$$

and consumption allocations in period 0

$$\begin{aligned}c_0^y &= \frac{1}{1 + \beta \omega_{-1}} (1 - s_0^{SP} (1 - \kappa) (1 - \alpha)) k_0^\alpha \\ c_0^o &= \frac{\beta \omega_{-1}}{1 + \beta \omega_{-1}} (1 - s_0^{SP} (1 - \kappa) (1 - \alpha)) k_0^\alpha\end{aligned}$$

and in all periods $t > 0$ by a constant saving rate

$$s^{SP} = \frac{k_{t+1}}{(1 - \kappa) (1 - \alpha) k_t^\alpha} = \frac{\alpha \theta}{(1 - \kappa) (1 - \alpha)}$$

and associated sequence of capital stocks

$$k_{t+1} = \alpha \theta k_t^\alpha$$

and consumption levels

$$\begin{aligned} c_t^y &= \frac{\theta(1 - \alpha\theta)k_t^\alpha}{\theta + \beta} \\ c_t^o &= \frac{\beta(1 - \alpha\theta)k_t^\alpha}{\theta + \beta} \end{aligned}$$

Corollary 5. *If $\theta = 1$ (associated with maximizing steady state utility), then the social planner chooses the golden rule saving rate*

$$s^{SP} = s^{GR} = \frac{\alpha}{(1 - \kappa)(1 - \alpha)}$$

and the capital stock converges, in the long run, to

$$k^{GR} = \alpha^{\frac{1}{1-\alpha}}$$

which satisfies

$$\alpha [k^{GR}]^{\alpha-1} = 1$$

and associated consumption levels

$$\begin{aligned} c^y &= \frac{(1 - \alpha)}{1 + \beta} \alpha^{\frac{\alpha}{1-\alpha}} \\ c_t^o &= \frac{\beta(1 - \alpha)}{1 + \beta} \alpha^{\frac{\alpha}{1-\alpha}} \end{aligned}$$

The social planner chooses the golden rule capital stock k^{GR} maximizing net output $y^{GR} = (k^{GR})^\alpha - k^{GR}$ and splits it efficiently between c^y and c^o according to the rule $c^o = \beta c^y$.

Obviously, the Ramsey equilibrium is not Pareto efficient because it does not provide full consumption insurance against idiosyncratic income risk. What is more remarkable is that even though the optimal Ramsey saving rate is independent of income risk (and the same as in a model where income risk is absent), it is in general different from the saving rate optimally chosen by the social planner (who fully insures the idiosyncratic income risk). This result is summarized in the next

Corollary 6. *For a fixed social discount factor $\theta \in [0, 1]$, the optimal Ramsey saving rate equals the saving rate chosen by the social planner if and only if the following knife edge*

condition is satisfied:

$$(1 - \kappa) = \frac{\theta(1 + \alpha\beta)}{(1 - \alpha)(\beta + \theta)}$$

Note that the Ramsey government can surely implement the saving rate desired by the social planner through an appropriate choice of taxes, but unless the condition above is satisfied, it is suboptimal to do so. The reason is that the Ramsey government has no instruments to transfer resources across generations and thus forcing the planner saving rate onto households (by appropriate choice of the capital tax rate) results in an equilibrium allocation of consumption across the young and the old that is typically suboptimal.³¹

E.2 Proof of Constrained Efficiency of Ramsey Allocation

Proof. Define the saving rate of the constrained planner as

$$s_t = \frac{k_{t+1}}{(1 - \kappa)MPL(k_t)} = \frac{k_{t+1}}{(1 - \alpha)(1 - \kappa)k_t^\alpha}$$

Thus, the law of motion for the effective capital stock for the constrained planner is

$$k_{t+1} = s_t(1 - \alpha)(1 - \kappa)k_t^\alpha$$

as in the Ramsey problem. Furthermore, from the constraints on the constrained planner

$$\begin{aligned} c_t^y &= (1 - \kappa)MPL(k_t) - k_{t+1} = (1 - s_t)(1 - \kappa)(1 - \alpha)k_t^\alpha \\ c_{t+1}^o(\eta_{t+1}) &= k_{t+1}MPK(k_{t+1}) + \kappa\eta_{t+1}MPL(k_{t+1}) \\ &= \alpha k_{t+1}^\alpha + \kappa\eta_{t+1}(1 - \alpha)k_{t+1}^\alpha \\ &= [\alpha + \kappa\eta_{t+1}(1 - \alpha)]k_{t+1}^\alpha. \end{aligned}$$

Thus consumption is the same as in the Ramsey equilibrium and the solution, in terms of saving rates, of the constrained planner problem is the same as the Ramsey equilibrium. \square

³¹Finally note that if one were to treat the social discount factor θ as a *free parameter*, then one concludes that the Ramsey optimal saving rate is efficient, in that it is identical to the choice of the social planner with a *different social discount rate* $\theta^{SP} = \frac{(\beta + \theta)(1 - \kappa)(1 - \alpha)}{1 + \alpha\beta}$.

E.3 Proof of Pareto-Improving Tax-Induced Transition

E.3.1 Log Utility

Proof of Proposition 6. The capital stock evolves according to

$$k_{t+1} = s_t(1 - \kappa)(1 - \alpha)k_t^\alpha.$$

Therefore if the Ramsey government implements s^* through positive capital taxes in the first period of the transition this will lead to a falling capital stock along the transition. Recall from (1) that utility of a generation born in period t is given by

$$V_t = \ln(c_t^y) + \beta \int \ln(c_{t+1}^o(\eta_{t+1}))d\Psi.$$

Now, suppose that the policy is implemented (as a surprise) in period 1 where $k_1 = k_0$. The initial old are unaffected by this policy and thus indifferent to the tax reform. Now we need to characterize the utility consequences for all generations born along the transition. Denoting by $s_0 = s^{CE}$ the equilibrium saving rate in the initial steady state, we have

$$\Delta V_t = V_t(s^*) - V_t(s_0) = \ln(c_t^y(s^*)) - \ln(c_t^y(s_0)) + \beta \int (\ln(c_{t+1}^o(s^*)) - \ln(c_{t+1}^o(s_0))) d\Psi.$$

where the consumption allocations are

$$\begin{aligned} c_t^y(s_t) &= (1 - s_t)(1 - \kappa)(1 - \alpha)k_t^\alpha \\ c_{t+1}^o(\eta_{t+1}; s_t) &= s_t(1 - \kappa)(1 - \alpha)k_t^\alpha \alpha k_{t+1}^{\alpha-1} + \kappa \eta_{t+1}(1 - \alpha)k_{t+1}^\alpha \\ &= [\alpha + \kappa \eta_{t+1}(1 - \alpha)] k_{t+1}^\alpha. \end{aligned}$$

Thus

$$\Delta V_t = \underbrace{\ln[(1 - s^*)k_t^\alpha] - \ln[(1 - s_0)k_0^\alpha]}_{=\Delta V_t^+} + \alpha\beta\Gamma_2 \left(\underbrace{\ln[k_{t+1}] - \ln[k_0]}_{=\Delta V_t^-} \right)$$

Since the capital stock is monotonically decreasing along the transition, $\Delta V_t^- < 0$ for all $t > 0$ and $\Delta V_s^- < \Delta V_t^- < 0$ for all $s > t > 0$, and we call ΔV_t^- the “loss” term. From the monotonically decreasing capital stock it also follows that ΔV_t^+ is monotonically decreasing along the transition. Since in the limit we have $\lim_{t \rightarrow \infty} \Delta V_t > 0$ (because s^*

maximizes steady state utility), it follows that $\Delta V_t^+ > 0$ for all $t > 0$ and we therefore refer to ΔV_t^+ as the “gains” term. Finally, since gains are monotonically decreasing and losses—the absolute value $|V_t^-|$ —are monotonically increasing we achieve the smallest gains and largest losses for $t \rightarrow \infty$ and since $\lim_{t \rightarrow \infty} \Delta V_t > 0$, it follows that $\Delta V_t > 0$ in all $t > 0$. \square

E.3.2 Generalization

Consider the additively separable life-time utility function V_t as

$$V_t = u(c_t^y) + g(c_{t+1}^o, \Psi) \quad (37)$$

with $u' > 0, u'' < 0$ for all $c_t^y > 0$ and $g' > 0, g'' < 0$ for all $c_{t+1}^o > 0$. Aggregating second period consumption with function $g(\cdot)$ nests standard (discounted) expected utility formulations as well as non-expected utility preferences such as Epstein-Zin-Weil preferences, analyzed in Section 6. As before, write consumption allocations in terms of the saving rate s as $(c_t^y(s), c_{t+1}^o(\eta, s))$. As shorthand, below we denote as $u_s = u'(c_t^y(s)) \times c_t^y(s)'$, with g_s defined correspondingly. Given this notation the first-order condition of the Ramsey problem for $\theta = 1$ is

$$\frac{\partial V_\infty}{\partial s} = u_s + g_s = 0 \quad \Leftrightarrow \quad -u_s = g_s. \quad (38)$$

We make the following additional

Assumption 7.

$$\lim_{s \rightarrow 1} -u_s > \lim_{s \rightarrow 1} g_s \quad (39)$$

and, for all $s \in (\alpha, 1)$,

$$\varepsilon_{u',c} = -\frac{u''(c_t^y(s))}{u'(c_t^y(s))} < \frac{c_t^y(s)''}{c_t^y(s)'} = \varepsilon_{c_s,s}, \quad (40)$$

where $\varepsilon_{u',c}$ is the semi-elasticity of marginal utility³² with respect to consumption c^y and $\varepsilon_{c_s,s}$ is the semi-elasticity of consumption c^y with respect to the saving rate s .

³²In a static stochastic environment this would be equal to the measure of absolute risk aversion. We prefer the term semi-elasticity of marginal utility because first period consumption is not stochastic.

The next proposition generalizes Proposition 6 to additively separable utility functions with the above properties. It also provides conditions for existence and uniqueness of a solution to (38):

Proposition 20. *Let the utility function be given by (37). Under assumption 7 the solution to (38) gives a unique $s^* \in (\alpha, 1)$. Further assume that $s^{CE} > s^*$. Then implementing s^* in period $t = 0$ for all $t \geq 0$ leads to a Pareto improving transition.*

Before proving the above proposition, note that condition (39) is required for existence, and condition (40) for uniqueness of $s^* \in (\alpha, 1)$. We further show that condition (40) implies for $s^{CE} > s^*$ that $\frac{\partial V_\infty}{\partial s} < 0$ so that the generation born in the limit of the transition when the economy approaches the new steady state benefits from implementing $s^* < s^{CE}$. We later establish for Epstein-Zin-Weil preferences, which nest CRRA preferences as a special case, that all these conditions are satisfied. Thus, we show analytically that the conditions apply quite generally. For the general class of HARA utility functions

$$u(c) = \frac{1-\gamma}{\gamma} \left(\frac{\iota \cdot c}{1-\gamma} + \xi \right)^\gamma$$

with parameters $\iota > 0, \xi, \gamma$, and the restriction $\frac{\iota \cdot c}{1-\gamma} + \xi > 0$ and $\gamma \neq 1$ (ruling out linear utility) condition (39) may fail to hold so that there is no solution to the Ramsey problem. For instance, with exponential utility condition (39) may fail to hold since there is no lower Inada condition so that $\lim_{s \rightarrow 1} -u_s < \infty$.³³

As for the assumption that $s^{CE} > s^*$ notice that we earlier established that s^{CE} is increasing with risk if there is precautionary savings. Thus, with sufficient risk we have $s^{CE} > s^*$. Also, as for the second part of the proposition on the Pareto improving transition, the proof follows exactly the same logic as the proof of Proposition 6.

We do not address in this proposition whether the economy is dynamically efficient. Of course, as before, the interesting case is where $s^* < s^{CE} < s^{GR}$, where $s^{GR} = \frac{\alpha}{(1-\alpha)(1-\kappa)}$ is the golden rule saving rate. Finally, notice that the proposition is silent about implementation. We address implementation under the assumption of existence of a unique s^* in the subsequent Proposition 21 for expected utility and later in Proposition 23 for EZW utility.

³³Consider nested exponential utility, i.e., $\gamma = -\infty$, and $\xi = 1$. Further parameterize $\iota = 1, \alpha = 0.33, \kappa = 0.7$ and $\eta = 1$, i.e., a degenerate deterministic case. Also assume an expected utility formulation with $\beta = 1$

$$g(c^o; \Psi) = \beta \int u(c^o(\eta)) d\Psi(\eta).$$

Then condition (39) fails to hold, an interior s^* does not exist and the optimal saving rate is $s^* = 1$.

Proof of Proposition 20. First, we establish that s^* is unique and that with uniqueness we get for $s^{CE} > s^*$ that $\frac{\partial V_\infty}{\partial s} < 0$. To show this, we analyze the first-order condition of the Ramsey government (38). The next steps will establish that (i) $g_s > 0$ is a continuous and downward sloping function in s , (ii) $-u_s > 0$ for $s > \alpha$, and (iii) that condition (40) is required for a single crossing of g_s and $-u_s$. Findings (i)-(ii) together with (39) establish existence, the additional finding (iii) uniqueness of s^* .

Start from the allocation in the long-run steady state. Recall from Section E.3 above that consumption when young and old is

$$\begin{aligned} c_t^y &= (1 - s_t)(1 - \kappa)(1 - \alpha)k_t^\alpha, \\ c_{t+1}^o &= (\alpha + \kappa(1 - \alpha)\eta) k_{t+1}^\alpha, \end{aligned}$$

where

$$k_{t+1} = s_t(1 - \kappa)(1 - \alpha)k_t^\alpha. \quad (41)$$

In steady state we thus have

$$k = (s(1 - \kappa)(1 - \alpha))^{\frac{1}{1-\alpha}}$$

and therefore steady state consumption allocations are

$$c^y = (1 - s)s^{\frac{\alpha}{1-\alpha}} ((1 - \alpha)(1 - \kappa))^{\frac{1}{1-\alpha}} \quad (42a)$$

$$c^o = (\alpha + \kappa(1 - \alpha)\eta) ((1 - \alpha)(1 - \kappa))^{\frac{\alpha}{1-\alpha}} s^{\frac{\alpha}{1-\alpha}}. \quad (42b)$$

Use this in the social welfare function for $\theta = 1$ to obtain

$$\begin{aligned} V_\infty &= u(c^y) + g(c^o; \Psi) \\ &= u\left((1 - s)s^{\frac{\alpha}{1-\alpha}} ((1 - \alpha)(1 - \kappa))^{\frac{1}{1-\alpha}}\right) + g((\alpha + \kappa(1 - \alpha)\eta) ((1 - \alpha)(1 - \kappa))^{\frac{\alpha}{1-\alpha}} s^{\frac{\alpha}{1-\alpha}}; \Psi). \end{aligned}$$

From the above we readily observe that $g_s > 0$ as well as $g_{ss} < 0$ because of decreasing

marginal utility.³⁴ To establish existence of s^* observe that

$$u_s = u'(c^y(s)) \times c^y(s)' = u'((1-\alpha)(1-\kappa))^{\frac{1}{1-\alpha}} \left(-1 + \frac{\alpha}{1-\alpha}(1-s)s^{-1} \right) s^{\frac{\alpha}{1-\alpha}}$$

$$< 0 \quad \Leftrightarrow \quad c^y(s)' < 0 \quad \Leftrightarrow \quad s > \alpha.$$

because $u'(c^y(s)) > 0$ and thus $u_s < 0$ for $s > \alpha$. If, in addition, condition (39) holds, then there exists at least one solution $s^* \in (\alpha, 1)$. Also notice that condition (39) holds if u satisfies the Inada condition, because then $\lim_{s \rightarrow 1} -u_s = \infty$ and $\lim_{s \rightarrow 1} g_s < \infty$.

To establish uniqueness we further require that $u'' < 0$ for all $s \in (\alpha, 1)$ so that $-u_s$ is continuously upward sloping. Observe that

$$u_{ss} = u''(c^y) c^y(s)' + u'(c^y) c^y(s)'' < 0 \quad \Leftrightarrow \quad \varepsilon_{u',c} = -\frac{u''(c^y)}{u'(c^y)} < \frac{c^y(s)''}{c^y(s)'} = \varepsilon_{c^y,s}$$

which limits the (positive) semi-elasticity of marginal utility $\varepsilon_{u',c}$ from above. For the semi-elasticity of consumption $\varepsilon_{c^y,s}$ notice that we have already established that $c^y(s)' < 0$ for $s \in (\alpha, 1)$. We next show that for $s \in (\alpha, 1)$ also $c^y(s)'' < 0$ so that $\varepsilon_{c^y,s} > 0$. To see this, write

$$c^y(s)'' = ((1-\alpha)(1-\kappa))^{\frac{1}{1-\alpha}} \frac{\alpha}{1-\alpha} s^{\frac{\alpha}{1-\alpha}-1} \left[-2 + (1-s) \frac{2\alpha-1}{1-\alpha} s^{-1} \right]$$

and thus $c^y(s)'' < 0$ if

$$-2 + (1-s) \frac{2\alpha-1}{1-\alpha} s^{-1} < 0 \quad \Leftrightarrow \quad s > 2\alpha - 1$$

Before, we have shown that for $s > \alpha$ we have $c^y(s)' < 0$ and since $\alpha > 2\alpha - 1 \Leftrightarrow \alpha < 1$ we know that $s > \alpha$ implies that $c^y(s)'' < 0$ and thus for $s \in (\alpha, 1)$ we get $\frac{c^y(s)''}{c^y(s)'} > 0$. Also, since by property (40) the function $-u_s$ is continuous and upward sloping and since g_s is downward sloping we have that if $s^* \in (\alpha, 1)$ exists, then $s^{CE} > s^*$ implies that $V_\infty'(s) < 0$.

Along the transition, recall that the consumption allocations for generation t is

$$c_t^y(s_t) = (1-s_t)(1-\kappa)(1-\alpha)k_t^\alpha$$

$$c_{t+1}^o(\eta_{t+1}; s_t) = [\alpha + \kappa\eta_{t+1}(1-\alpha)] k_{t+1}^\alpha.$$

³⁴Specifically, we have assumed that $g_c > 0$, $g_{cc} < 0$. Observe from (42b) that $c^o(s)' > 0$ so that $g_s > 0$ and $g_{ss} < 0$.

Thus, assuming a unique $s^* < s^{CE}$ we get

$$\Delta V_t = \underbrace{u((1-s^*)(1-\kappa)(1-\alpha)k_t^\alpha) - u((1-s^{CE})(1-\kappa)(1-\alpha)k_0^\alpha)}_{=\Delta V_t^+} + \underbrace{g([\alpha + \kappa\eta_{t+1}(1-\alpha)]k_{t+1}^\alpha; \Psi) - g([\alpha + \kappa\eta_{t+1}(1-\alpha)]k_0^\alpha; \Psi)}_{=\Delta V_t^-}$$

and since $\frac{\partial c_t^y}{\partial k_t} > 0$ as well as $\frac{\partial c_{t+1}^o}{\partial k_{t+1}} > 0$ the same arguments on the behavior of V_t^+ and V_t^- along the transition as in the proof of Proposition 6 apply. \square

E.3.3 Implementation

Observe that the proof above does not say anything about implementation of the saving rates though taxation of capital. The next proposition contains a fairly general implementation result for expected utility. Proposition 23 extends this result to EZW utility.

Proposition 21. *If the utility function in both periods is of the HARA form,*

$$u(c) = \frac{1-\gamma}{\gamma} \left(\frac{\iota c}{1-\gamma} + \xi \right)^\gamma, \quad (43)$$

with parameters $\iota > 0, \xi, \gamma, \gamma \neq 1$ such that $\frac{\iota c}{1-\gamma} + \xi > 0$, then in general equilibrium the saving rate s is strictly decreasing in the tax rate τ so $s^ \in (\alpha, 1]$ can be implemented by a unique (typically time-dependent) tax rate τ_{t+1}^* .*

Proof. Start from the Euler equation given current period t aggregate wages $w_t = (1-\alpha)k_t^\alpha$

$$\begin{aligned} u'[(1-\kappa)w_t(1-s(\tau_{t+1}))] = \\ \alpha\beta(1-\tau_{t+1})((1-\kappa)w_t)^{\alpha-1}s(\tau_{t+1})^{\alpha-1} \int u'[(\alpha + (1-\alpha)\kappa\eta)[(1-\kappa)w_t]^\alpha s(\tau_{t+1})^\alpha] d\Psi(\eta). \end{aligned} \quad (44)$$

Totally differentiate (44) to get

$$\begin{aligned}
& - (1 - \kappa)w_t u'' [(1 - \kappa)w_t(1 - s(\tau_{t+1}))] \frac{ds(\tau_{t+1})}{d\tau_{t+1}} = \alpha\beta ((1 - \kappa)w_t)^{\alpha-1} \\
& \left[-s(\tau_{t+1})^{\alpha-1} + (1 - \tau_{t+1})(\alpha - 1)s(\tau_{t+1})^{\alpha-2} \frac{ds(\tau_{t+1})}{d\tau_{t+1}} \right] \int u' [(\alpha + (1 - \alpha)\kappa\eta) [(1 - \kappa)w_t]^\alpha s(\tau_{t+1})^\alpha] d\Psi(\eta) \\
& + \alpha^2\beta(1 - \tau_{t+1}) ((1 - \kappa)w_t)^{2\alpha-1} s(\tau_{t+1})^{2(\alpha-1)} \frac{ds(\tau_{t+1})}{d\tau_{t+1}} \\
& \cdot \int u'' [(\alpha + (1 - \alpha)\kappa\eta) [(1 - \kappa)w_t]^\alpha s(\tau_{t+1})^\alpha] (\alpha + (1 - \alpha)\kappa\eta) d\Psi(\eta).
\end{aligned}$$

Now use the notation

$$\begin{aligned}
c^y(s(\tau_{t+1})) &= (1 - \kappa)w_t(1 - s(\tau_{t+1})) \\
c^o(s(\tau_{t+1}), \eta) &= (\alpha + (1 - \alpha)\kappa\eta) [(1 - \kappa)w_t]^\alpha s(\tau_{t+1})^\alpha
\end{aligned}$$

and divide by $(1 - \kappa)w_t$ to rewrite this further as

$$\begin{aligned}
& - u'' [c^y(s(\tau_{t+1}))] \frac{ds(\tau_{t+1})}{d\tau_{t+1}} = -\alpha\beta ((1 - \kappa)w_t)^{\alpha-2} \\
& \left[s(\tau_{t+1})^{\alpha-1} + (1 - \tau_{t+1})(1 - \alpha)s(\tau_{t+1})^{\alpha-2} \frac{ds(\tau_{t+1})}{d\tau_{t+1}} \right] E [u' (c^o(s(\tau_{t+1}), \eta))] \\
& + \alpha^2\beta(1 - \tau_{t+1}) ((1 - \kappa)w_t)^{2(\alpha-1)} s(\tau_{t+1})^{2(\alpha-1)} \frac{ds(\tau_{t+1})}{d\tau_{t+1}} \\
& \cdot \int u'' [(\alpha + (1 - \alpha)\kappa\eta) [(1 - \kappa)w_t]^\alpha s(\tau_{t+1})^\alpha] (\alpha + (1 - \alpha)\kappa\eta) d\Psi(\eta).
\end{aligned}$$

Since $u' > 0$ and $u'' < 0$ ambiguity of implementation may come from the expression

$$\int u'' [(\alpha + (1 - \alpha)\kappa\eta) [(1 - \kappa)w_t]^\alpha s(\tau_{t+1})^\alpha] (\alpha + (1 - \alpha)\kappa\eta) d\Psi(\eta). \quad (45)$$

Before proceeding observe that without risk implementation is unambiguous since then

$$u'' [(\alpha + (1 - \alpha)\kappa) [(1 - \kappa)w_0]^\alpha s(\tau)^\alpha] (\alpha + (1 - \alpha)\kappa) < 0.$$

With income risk, observe that with HARA utility (43) we have

$$u' = \iota \left(\frac{\iota c}{1 - \gamma} + \xi \right)^{\gamma-1}, \quad u'' = -\iota^2 \left(\frac{\iota c}{1 - \gamma} + \xi \right)^{\gamma-2}$$

and thus (45) becomes

$$\begin{aligned} & - \int \iota^2 \left[\frac{\iota}{1 - \gamma} (\alpha + (1 - \alpha)\kappa\eta) [(1 - \kappa)w_t]^\alpha s(\tau_{t+1})^\alpha + \xi \right]^{\gamma-2} (\alpha + (1 - \alpha)\kappa\eta) d\Psi(\eta) \\ & = -\iota^2 \int \left[\left(\frac{\iota}{1 - \gamma} (\alpha + (1 - \alpha)\kappa\eta) [(1 - \kappa)w_t]^\alpha s(\tau_{t+1})^\alpha + \xi \right) (\alpha + (1 - \alpha)\kappa\eta)^{\frac{1}{\gamma-2}} \right]^{\gamma-2} d\Psi(\eta) \\ & = -\iota^2 \int \left[\left(\frac{\iota}{1 - \gamma} (\alpha + (1 - \alpha)\kappa\eta)^{\frac{\gamma-1}{\gamma-2}} [(1 - \kappa)w_t]^\alpha s(\tau_{t+1})^\alpha + \xi (\alpha + (1 - \alpha)\kappa\eta)^{\frac{1}{\gamma-2}} \right) \right]^{\gamma-2} d\Psi(\eta) \\ & = \Lambda(s(\tau_{t+1}); \iota, \xi, \alpha, \kappa, \gamma, \eta) < 0 \end{aligned}$$

and thus for HARA preferences defined in the proposition there is a strictly downward-sloping relationship between s_t and τ_{t+1} . \square

E.3.4 Marginal Reforms

The next corollary studies marginal tax reforms rather than implementing the full Ramsey equilibrium.

Corollary 7. *Let Assumption 7 hold and assume that $s^{CE} > s^*$. Implementing a saving rate $s^{CE} - \epsilon \geq s^*$ for small $\epsilon > 0$ in all periods $t \geq 0$ through a time-varying tax rate τ_{t+1} yields a Pareto improvement.*

Proof. Replace in the proof of Proposition 20 s^* by $s^{CE} - \epsilon \geq s^*$ to note that the same arguments on monotone transitions of the gains and loss terms apply. \square

E.4 Savings Subsidy Does Not Induce Pareto Improvement

In this section we show, in contrast to the previous section, that even if $s^{CE} < s^*$, implementing the Ramsey (for $\theta = 1$) saving rate s^* through a savings subsidy $\tau^* < 0$ does *not* lead to a Pareto improving transition. Exploiting the fact that in the first period of the transition the capital stock $k_1 = k_0$ is predetermined, and the capital stock in $t = 2$ satisfies

$$k_2 = s(1 - \alpha)(1 - \kappa)k_0^\alpha$$

for any saving rate implemented by a given tax policy. Thus we can calculate lifetime utility of the first transition generation, as a function of an implemented saving rate s , as

$$\begin{aligned} V_1(s) &= \ln((1-s)(1-\kappa)(1-\alpha)k_0^\alpha) + \beta \int \ln(\alpha + \kappa\eta_2(1-\alpha)) (s(1-\alpha)(1-\kappa)k_0^\alpha)^\alpha d\Psi(\eta) \\ &= \ln(1-s) + \beta\alpha \ln(s) + \ln((1-\kappa)(1-\alpha)k_0^\alpha) \\ &\quad + \beta \int \ln(\alpha + \kappa\eta_2(1-\alpha)) ((1-\alpha)(1-\kappa)k_0^\alpha)^\alpha d\Psi(\eta) \end{aligned}$$

and thus

$$\begin{aligned} V_1'(s) &= -\frac{1}{1-s} + \frac{\alpha\beta}{s} \\ V_1''(s) &= -\frac{1}{(1-s)^2} - \frac{\alpha\beta}{s^2} < 0 \end{aligned}$$

and thus $V_1(s)$ is strictly concave in s . Therefore, if $V_1'(s = s^{CE}) \leq 0$, then $V(s = s^{CE}) > V(s)$ for all $s > s^{CE}$. We have

$$\begin{aligned} V_1'(s = s^{CE}) &= -\frac{1}{1-s^{CE}} + \alpha\beta \frac{1}{s^{CE}} \leq 0 \\ \Leftrightarrow \quad s^{CE} &\geq \frac{\alpha\beta}{1+\alpha\beta} \end{aligned}$$

which is satisfied, exploiting expression (12) for the optimal competitive equilibrium saving rate (with zero taxes). Thus not only is implementing $\tau^* < 0$ not Pareto improving if $s^{CE} < s^*$, but in fact any policy reform that induces a saving rate in period 1 above the competitive saving rate with zero taxes, s^{CE} , will not result in a Pareto improvement (since it will make the first generation strictly worse off).

F Analysis of General Epstein-Zin Utility

F.1 Competitive Equilibrium for Given Tax Policy

Household maximization delivers

$$1 = \beta(1 - \tau_{t+1})R_{t+1} \left[\int \left(\frac{c_{t+1}^o(\eta_{t+1})}{c_t^y} \right)^{1-\sigma} d\Psi(\eta_{t+1}) \right]^{\frac{\sigma - \frac{1}{\rho}}{1-\sigma}} \int \left(\frac{c_{t+1}^o(\eta_{t+1})}{c_t^y} \right)^{-\sigma} d\Psi(\eta_{t+1}).$$

and, using the expressions for consumption in both periods and the law of motion of the capital stock, as in the previous analysis we can rewrite the first-order condition as

$$1 = \alpha\beta ((1 - \kappa)(1 - \alpha))^{(\alpha-1)(1-\frac{1}{\rho})} (1 - \tau_{t+1}) k_t^{\alpha(\alpha-1)(1-\frac{1}{\rho})} s_t^{(\alpha-1)(1-\frac{1}{\rho})} \left(\frac{1 - s_t}{s_t} \right)^{\frac{1}{\rho}} \tilde{\Gamma}.$$

In steady state the Euler equation reads as

$$1 = \alpha\beta ((1 - \kappa)(1 - \alpha))^{(\alpha-1)(1-\frac{1}{\rho})} (1 - \tau) k^{\alpha(\alpha-1)(1-\frac{1}{\rho})} s^{(\alpha-1)(1-\frac{1}{\rho})} \left(\frac{1 - s}{s} \right)^{\frac{1}{\rho}} \tilde{\Gamma}$$

where

$$k = [(1 - \kappa)(1 - \alpha)s]^{\frac{1}{1-\alpha}}$$

is the steady state capital stock. Inserting it into the Euler equation delivers

$$1 = (1 - \tau)\alpha\beta ((1 - \kappa)(1 - \alpha))^{(\frac{1}{\rho}-1)} \frac{(1 - s)^{\frac{1}{\rho}}}{s} \tilde{\Gamma} \quad (46)$$

where $\tilde{\Gamma}$ is defined in the main text. This result is the generalization of the log-case where $\rho = \sigma = 1$, and where the Euler equation was given as

$$1 = (1 - \tau)\alpha\beta \left(\frac{1 - s}{s} \right) \Gamma$$

with Γ defined as in the main text. Thus our previous analysis for log-utility is just a special case. Also note that if $\rho = 1$ but $\sigma \neq 1$, then the steady state Euler equation is given by

$$1 = (1 - \tau)\alpha\beta \left(\frac{1 - s}{s} \right)^{\frac{1}{\rho}} \tilde{\Gamma}$$

but

$$\tilde{\Gamma} = \frac{\int (\alpha + (1 - \alpha)\kappa\eta)^{-\sigma} d\Psi(\eta)}{[\int (\alpha + (1 - \alpha)\kappa\eta)^{1-\sigma} d\Psi(\eta)]} \neq \int (\kappa\eta(1 - \alpha) + \alpha)^{-1} d\Psi(\eta) = \Gamma_{\sigma=1}$$

F.1.1 Precautionary Savings Behavior in the Competitive Equilibrium

In order to aid with the interpretation of the optimal Ramsey tax rate it is useful to establish conditions under which, for a fixed tax rate, the saving rate in competitive equilibrium is

increasing in income risk.

Proposition 22. *If $\tilde{\Gamma}$ is strictly increasing in income risk, then for any given tax rate $\tau \in (-\infty, 1)$ the steady state saving rate $s^{CE}(\tau)$ in competitive equilibrium is strictly increasing in income risk. If $\tilde{\Gamma}$ is strictly decreasing in income risk, then so is $s^{CE}(\tau)$.*

Proof. Rewrite equation (46) as

$$f(s) = (1 - \tau)\alpha\beta((1 - \kappa)(1 - \alpha))^{(\frac{1}{\rho}-1)} \frac{(1 - s)^{\frac{1}{\rho}}}{s} - \frac{1}{\tilde{\Gamma}}.$$

Then a saving rate $s^{CE}(\tau)$ that satisfies $f(s^{CE}(\tau)) = 0$ is a steady state equilibrium saving rate. We readily observe that f is continuous and strictly decreasing in s , with

$$\begin{aligned} \lim_{s \rightarrow 0} f(s) &= \infty \\ f(1) &= -\frac{1}{\tilde{\Gamma}} < 0 \end{aligned}$$

and thus for each $\tau \in (-\infty, 1)$ there is a unique $s = s^{CE}(\tau)$ that satisfies $f(s^{CE}(\tau)) = 0$. Inspection of f immediately reveals that $s^{CE}(\tau)$ is strictly increasing in $\tilde{\Gamma}$, from which the comparative statics results follow. \square

Corollary 8. *For any given $\tau \in (-\infty, 1)$, the steady state saving rate $s^{CE}(\tau)$ increases in income risk if either $\rho \leq 1$, or $1 < \rho < \frac{1}{\sigma}$.*

Proof. Follows directly from the previous proposition and Lemma 1 in the main text (and proved in the next section) characterizing the behavior of $\tilde{\Gamma}$ with respect to income risk. \square

Proposition 22 establishes a sufficient condition for the private saving rate to increase in income risk. But, for $\rho > \frac{1}{\sigma} > 1$ it is possible that the combination of individual savings behavior and general equilibrium factor price movements lead to the result that, for *fixed* government policy, the equilibrium saving rate is decreasing in income risk.³⁵ We will show below that this in turn is a necessary condition for the optimal Ramsey tax rate to decrease in income risk.

³⁵Also observe that a parameter constellation $1 < \rho < \frac{1}{\sigma}$ pairs a high IES with a preference for a *late* resolution of risk in a multi-period (more than two periods) model. Interestingly, the competitive equilibrium saving rate may therefore decrease in income risk precisely when we pair a high IES with a preference constellation for *early* resolution of risk.

F.2 Ramsey Problem for Unit IES

Now we use the formulation of lifetime utility in equation (24). Then it is straightforward to show that for $\rho = 1$ the analysis of the Ramsey problem proceeds exactly as for log utility

$$\begin{aligned}
W(k) &= \Theta_0 + \Theta_1 \ln(k) \\
&= \max_{s \in [0,1]} \left\{ \ln((1-s)(1-\kappa)(1-\alpha)k^\alpha) \right. \\
&\quad \left. + \frac{\beta}{1-\sigma} \ln \int (\kappa \eta w(s) + R(s)s(1-\kappa)(1-\alpha)k^\alpha)^{1-\sigma} d\Psi(\eta) + \theta W(k') \right\} \\
&= \max_{s \in [0,1]} \left\{ \ln((1-s)(1-\kappa)(1-\alpha)k^\alpha) \right. \\
&\quad \left. + \frac{\beta}{1-\sigma} \ln \int ([\kappa \eta(1-\alpha) + \alpha] [s(1-\kappa)(1-\alpha)k^\alpha]^\alpha)^{1-\sigma} d\Psi(\eta) + \theta W(s(1-\kappa)(1-\alpha)k^\alpha) \right\} \\
&= \alpha [1 + \theta \Theta_1 + \alpha \beta] \ln(k) + \ln[(1-\kappa)(1-\alpha)] + \theta \Theta_0 + \theta \Theta_1 \ln((1-\kappa)(1-\alpha)) \\
&\quad + \beta \alpha \ln[(1-\kappa)(1-\alpha)] + \frac{\beta \ln \int [\kappa \eta(1-\alpha) + \alpha]^{1-\sigma} d\Psi(\eta)}{1-\sigma} \\
&\quad + \max_{s \in [0,1]} \{ \ln(1-s) + \alpha \beta \ln(s) + \theta \Theta_1 \ln(s) \}
\end{aligned}$$

and the FOC delivers the optimal saving rate as in the main text:

$$s = \frac{\alpha(\beta + \theta)}{1 + \alpha\beta}.$$

These results clarify that the closed form solution, and the fact that the optimal saving rate is constant over time and independent of the level of capital, is driven by an $IES = \rho = 1$ (and obtained for arbitrary risk aversion), whereas the size of the capital tax needed to implement the optimal Ramsey allocation does depend on risk aversion σ , see Section F.1.1.

F.3 Steady State Analysis for Arbitrary IES

The Ramsey government seeking to maximize steady state lifetime utility has the objective function:

$$\begin{aligned}
V(s) &= \frac{(c_t^y)^{1-\frac{1}{\rho}} + \beta \left\{ \left[\int c_{t+1}^o (\eta_{t+1})^{1-\sigma} d\Psi \right]^{\frac{1}{1-\sigma}} \right\}^{1-\frac{1}{\rho}}}{1 - \frac{1}{\rho}} \\
&= \frac{((1-\kappa)(1-s)(1-\alpha)k^\alpha)^{1-\frac{1}{\rho}}}{1 - \frac{1}{\rho}} \\
&\quad + \frac{\beta [s(1-\kappa)(1-\alpha)k^\alpha]^{\alpha(1-\frac{1}{\rho})} \left\{ \left[\int \{[\kappa\eta(1-\alpha) + \alpha]\}^{1-\sigma} d\Psi \right]^{\frac{1}{1-\sigma}} \right\}^{1-\frac{1}{\rho}}}{1 - \frac{1}{\rho}} \\
&= \frac{((1-\kappa)(1-\alpha))^{1-\frac{1}{\rho}}}{1 - \frac{1}{\rho}} (1-s)^{(1-\frac{1}{\rho})} k^{\alpha(1-\frac{1}{\rho})} + \frac{\beta [(1-\kappa)(1-\alpha)]^{\alpha(1-\frac{1}{\rho})} \tilde{\Gamma}_2}{1 - \frac{1}{\rho}} s^{\alpha(1-\frac{1}{\rho})} k^{\alpha^2(1-\frac{1}{\rho})}
\end{aligned}$$

where

$$\tilde{\Gamma}_2 = \left[\int \{[\kappa\eta(1-\alpha) + \alpha]\}^{1-\sigma} d\Psi \right]^{\frac{1-\frac{1}{\rho}}{1-\sigma}} = \Gamma_2^{\frac{\sigma-\frac{1}{\rho}}{1-\sigma}} \Gamma_2.$$

Exploiting that in steady state

$$k = ((1-\kappa)(1-\alpha)s)^{\frac{1}{1-\alpha}}$$

yields

$$\begin{aligned}
V(s) &= \frac{((1-\kappa)(1-\alpha))^{1-\frac{1}{\rho}}}{1 - \frac{1}{\rho}} (1-s)^{(1-\frac{1}{\rho})} ((1-\kappa)(1-\alpha)s)^{\frac{\alpha(1-\frac{1}{\rho})}{1-\alpha}} \\
&\quad + \frac{\beta [(1-\kappa)(1-\alpha)]^{\alpha(1-\frac{1}{\rho})} \tilde{\Gamma}_2}{1 - \frac{1}{\rho}} (s)^{\alpha(1-\frac{1}{\rho})} ((1-\kappa)(1-\alpha)s)^{\frac{\alpha^2(1-\frac{1}{\rho})}{1-\alpha}} \\
&= \tilde{\phi} \left((1-s)^{(1-\frac{1}{\rho})} + \beta \tilde{\zeta} \tilde{\Gamma}_2 \right) s^{\frac{\alpha(1-\frac{1}{\rho})}{1-\alpha}},
\end{aligned}$$

where

$$\begin{aligned}\tilde{\phi} &= \frac{((1-\kappa)(1-\alpha))^{\frac{1-\frac{1}{\rho}}{1-\alpha}}}{1-\frac{1}{\rho}} \\ \tilde{\zeta} &= \left(\frac{1}{(1-\kappa)(1-\alpha)} \right)^{(1-\frac{1}{\rho})} > 0 \\ \tilde{\Gamma}_2 &= \left(\left[\int \{[\kappa\eta(1-\alpha) + \alpha]\}^{1-\sigma} d\Psi \right]^{\frac{1}{1-\sigma}} \right)^{1-\frac{1}{\rho}} > 0.\end{aligned}$$

Thus the steady state analysis in the main text carries through to Epstein-Zin-Weil utility almost entirely unchanged, but with the constant that maps earnings risk into the optimal saving rate now being affected both by risk aversion and the IES.

Hence, the optimal steady state saving rate is defined implicitly as

$$\frac{s}{(1-s)^{\frac{1}{\rho}}} = \frac{\alpha}{1-\alpha}(1-s)^{(1-\frac{1}{\rho})} + \beta \frac{\alpha}{1-\alpha} \tilde{\zeta} \tilde{\Gamma}_2 \quad (47)$$

and rewriting this equation yields

$$LHS(s) = s = \frac{\alpha}{1-\alpha} \left[(1-s) + \beta \tilde{\zeta} \tilde{\Gamma}_2 (1-s)^{\frac{1}{\rho}} \right] = RHS(s). \quad (48)$$

We observe that the left hand side is linearly increasing in s , with $LHS(0) = 0$ and $LHS(1) = 1$ and the right hand side is strictly decreasing in s , with $RHS(0) > 0$ and $RHS(1) = 0$. Since both sides are continuous in s , from the intermediate value theorem it follows that there is a unique $s^* \in (0, 1)$ solving the first order condition of the Ramsey problem (48). Since $RHS(s)$ is strictly increasing in $\tilde{\Gamma}_2$, the Ramsey saving rate is strictly increasing in $\tilde{\Gamma}_2$. The comparative statics of s^* with respect to income risk in the main text then directly follow from the properties of $\tilde{\Gamma}_2$ stated in Lemma 1.

For future reference we rewrite equation (48) as

$$\frac{(1-s)^{\frac{1}{\rho}}}{s} = \frac{\frac{1-\alpha}{\alpha} - \frac{(1-s)}{s}}{\beta \tilde{\zeta} \tilde{\Gamma}_2} = \frac{\frac{1}{\alpha} - \frac{1}{s}}{\beta \tilde{\zeta} \tilde{\Gamma}_2}. \quad (49)$$

F.4 Implementation

F.4.1 Implementation in Steady State

The optimal steady state capital tax rate τ^* satisfies, from equation (46)

$$1 = (1 - \tau^*)\alpha\beta((1 - \kappa)(1 - \alpha))^{\frac{1}{\rho}-1} \frac{(1 - s^*)^{\frac{1}{\rho}}}{s^*} \tilde{\Gamma}. \quad (50)$$

We observe that the optimal tax rate is strictly increasing in $\tilde{\Gamma}$ and strictly decreasing in the Ramsey saving rate s^* that is to be implemented. Further, recall that the Ramsey saving rate s^* itself satisfies the first order condition (49)

$$\frac{(1 - s^*)^{\frac{1}{\rho}}}{s^*} = \frac{\frac{1}{\alpha} - \frac{1}{s^*}}{\beta\zeta\tilde{\Gamma}_2} \quad (51)$$

and is impacted by income risk through $\tilde{\Gamma}_2$. Plugging (51) into (50) and exploiting the definition of $\tilde{\zeta}$ yields

$$1 = (1 - \tau^*) \left(1 - \frac{\alpha}{s^*}\right) \frac{\tilde{\Gamma}}{\tilde{\Gamma}_2}. \quad (52)$$

Lemma 1 establishes that $\frac{\tilde{\Gamma}}{\tilde{\Gamma}_2}$ is strictly increasing in income risk, and Proposition 9 in the main text establishes that an increase in income risk increases s^* if and only if $\rho < 1$ and decreases it if and only if $\rho > 1$. To sign the overall impact of income risk on the capital tax rate it is therefore useful to consider the following cases:

Case $\rho \leq 1$. This case gives clean results. From equation (52), since $\frac{\tilde{\Gamma}}{\tilde{\Gamma}_2}$ is strictly increasing in income risk, and since s^* is increasing in income risk for $\rho \leq 1$, strictly so if $\rho < 1$, it follows that τ^* is strictly increasing in risk.

Case $\rho > 1$ and $\sigma \leq 1/\rho$. In this case $\tilde{\Gamma}$ is strictly increasing in risk (Lemma 1) and s^* is strictly decreasing in risk (see Proposition 9) It then directly follows from equation (50) that τ^* is strictly increasing in income risk as well.

Case $\rho > 1$ and $\sigma > 1/\rho$. Since $\rho > 1$, the Ramsey saving rate s^* is strictly decreasing in income risk (which by itself calls for a tax rate that is strictly increasing in income risk), by equation (50). However, now the direct impact of income risk on taxes through the term $\tilde{\Gamma}$ might call for lower taxes since $\tilde{\Gamma}$ might now be decreasing in income risk. If $\tilde{\Gamma}$ is weakly

increasing in income risk, then so is τ^* . Thus a necessary condition for τ^* to decrease with income risk is for $\tilde{\Gamma}$ to be strictly decreasing with income risk. This in turn is a necessary and sufficient condition for the private saving rate in competitive equilibrium to decrease with income risk (see Proposition 22). Thus the Ramsey tax rate τ^* is strictly decreasing in income risk *only if* the private saving rate $s^{CE}(\tau)$ is strictly decreasing in income risk (for any given tax rate τ). The corresponding *if* statement is not necessarily true, as the numerical illustrations in the main text show.

Finally, one might conjecture that, since $\rho > 1$ and $\sigma > 1/\rho$ is required for the capital tax to decrease in income risk, that as long as both parameters are large enough the result will materialize. This conjecture turns out to be false, as an investigation of the most extreme case $\rho = \sigma = \infty$ shows. In this case lifetime utility is given by

$$V_t = c_t^y + \beta \underline{c}_{t+1}^o \quad (53)$$

where \underline{c}_{t+1}^o is consumption in old age if the lowest possible labor productivity realization $\eta = \underline{\eta}_{t+1}$ materializes. In this case one can solve analytically for the optimal interior Ramsey saving and tax rate, and show that the optimal tax rate is the higher the lower is $\underline{\eta}_{t+1}$ and thus the higher is income risk.³⁶

F.4.2 Implementation in Transition

Proposition 21 above provides a fairly general implementation result for expected utility. The next proposition extends this result to EZW utility. We use this result in our numerical analysis of Section 6.3.

Proposition 23. *If the utility function is of the EZW form, then in general equilibrium we have $s_\tau = \frac{\partial s_t}{\partial \tau_{t+1}} < 0$ and unambiguous implementation.*

Proof. Recall from Section F.1 that the first-order condition in any period t of the transition is

$$1 = \alpha\beta ((1 - \kappa)(1 - \alpha))^{(\alpha-1)(1-\frac{1}{\rho})} (1 - \tau_{t+1}) k_t^{\alpha(\alpha-1)(1-\frac{1}{\rho})} s_t^{(\alpha-1)(1-\frac{1}{\rho})} \left(\frac{1 - s_t}{s_t} \right)^{\frac{1}{\rho}} \tilde{\Gamma}.$$

³⁶In this case it is possible that the Ramsey government will want to implement a saving rate of $s = 1$ since households have linear preferences over consumption when young and minimum (across η) consumption when old. As long as $\underline{\eta}$ is sufficiently small, however, the Ramsey government prefers to implement an interior saving rate.

Observe that an increase in the tax rate decreases the RHS. Collect terms on the saving rate as

$$s_t^{(\alpha-1)(1-\frac{1}{\rho})} \left(\frac{1-s_t}{s_t} \right)^{\frac{1}{\rho}} = s_t^{(\alpha-1)(1-\frac{1}{\rho})-\frac{1}{\rho}} (1-s_t)^{\frac{1}{\rho}}$$

and notice that for any $\rho > 0$ term $(1-s_t)^{\frac{1}{\rho}}$ decreases in the saving rate. In response to an increase of the tax rate this force drives the saving rate down. To get unambiguous implementation, we thus require that the exponent

$$(\alpha-1) \left(1 - \frac{1}{\rho} \right) - \frac{1}{\rho} < 0 \Leftrightarrow \frac{1}{\rho} > 0 > 1 - \frac{1}{\alpha}$$

which holds for all $\alpha \in (0, 1)$. □

F.5 Decomposition of the FOC into $PE(s)$, $CG(s)$ and $FG(s)$

Proposition 24. *For $\theta = 1$, $\sigma \neq \frac{1}{\rho}$, terms $PE(s)$, $CG(s)$, $FG(s)$ are given by*

$$\begin{aligned} PE(s) &= -\frac{1}{1-s} \left(\frac{1-s}{s} \right)^{1-\frac{1}{\rho}} k(s)^{1-\frac{1}{\rho}} + \frac{\alpha\beta}{s} \tilde{\Gamma} k(s)^{\alpha(1-\frac{1}{\rho})} \\ CG(s) &= \frac{\alpha\beta}{s} k(s)^{\alpha(1-\frac{1}{\rho})} (\tilde{\Gamma}_2 - \tilde{\Gamma}) \\ FG(s) &= \frac{\alpha}{s(1-\alpha)} \left(\frac{1-s}{s} \right)^{1-\frac{1}{\rho}} k(s)^{1-\frac{1}{\rho}} + \frac{\alpha^2\beta}{s(1-\alpha)} k(s)^{\alpha(1-\frac{1}{\rho})} \tilde{\Gamma}_2 \end{aligned}$$

where $k(s) = (s(1-\kappa)(1-\alpha))^{\frac{1}{1-\alpha}}$ is the steady state capital stock.

Therefore,

$$PE(s) + CG(s) = -\frac{1}{1-s} \left(\frac{1-s}{s} \right)^{1-\frac{1}{\rho}} k(s)^{1-\frac{1}{\rho}} + \frac{\alpha\beta}{s} k(s)^{\alpha(1-\frac{1}{\rho})} \tilde{\Gamma}_2. \quad (54)$$

and

$$PE(s) + CG(s) + FG(s) = \left(\frac{1-s}{s} \right)^{1-\frac{1}{\rho}} k(s)^{1-\frac{1}{\rho}} \left(\frac{\alpha}{s(1-\alpha)} - \frac{1}{1-s} \right) + \frac{1}{s} k(s)^{\alpha(1-\frac{1}{\rho})} \frac{\alpha\beta}{(1-\alpha)} \tilde{\Gamma}_2. \quad (55)$$

Thus, compared to the expressions for these three effects we derived in Section 4.2, the partial equilibrium precautionary savings effect still cancels out the current generations general equilibrium effect ($\tilde{\Gamma}$ cancels out when adding up $PE(s)$ and $CG(s)$). However, additionally risk enters through $\tilde{\Gamma}_2$. With $\rho < 1$ an increase of risk increases $\tilde{\Gamma}_2$ thereby pushing up the desired saving rate of the Ramsey planner. The reason is that an increase of risk decreases the utility value of second period consumption of current generations (effect in $CG(s)$) and of all future generations (effect in $FG(s)$). With a low IES, it is optimal to compensate this with higher savings; vice versa for a high IES where the Ramsey planner rather prefers increased first-period consumption, respectively current generations consumption, over future consumption in response to an increase in risk.

Proof of Proposition 24. Calculating the respective terms yields

$$\begin{aligned}
PE(s) &= (1 - \kappa)(1 - \alpha)k^\alpha \left[-((1 - s)(1 - \kappa)(1 - \alpha)k^\alpha)^{-\frac{1}{\rho}} + \right. \\
&\quad \left. \alpha k'(s)^{\alpha-1} \beta \left(\int (\kappa\eta(1 - \alpha) + \alpha)^{1-\sigma} d\Psi \right)^{\frac{\sigma-\frac{1}{\rho}}{1-\sigma}} k'(s)^{\alpha(\sigma-\frac{1}{\rho})} \int (\kappa\eta(1 - \alpha) + \alpha)^{-\sigma} d\Psi k'(s)^{-\alpha\sigma} \right] \\
&= -\frac{1}{1-s} \left(\frac{1-s}{s} \right)^{1-\frac{1}{\rho}} k(s)^{1-\frac{1}{\rho}} + \frac{\alpha\beta}{s} \tilde{\Gamma} k(s)^{\alpha(1-\frac{1}{\rho})}.
\end{aligned}$$

and for

$$\begin{aligned}
CG(s) &= \beta \left(\int c^o(\eta)^{1-\sigma} d\Psi \right)^{\frac{\sigma-\frac{1}{\rho}}{1-\sigma}} \int (c^o(\eta)^{-\sigma}) [\kappa\eta w'(s) + (1 - \kappa)(1 - \alpha)k^\alpha R'(s)s] d\Psi(\eta) \\
&= \beta \Gamma_2^{\frac{\sigma-\frac{1}{\rho}}{1-\sigma}} k'(s)^{\alpha(\sigma-\frac{1}{\rho})} \int (\kappa\eta(1 - \alpha) + \alpha)^{-\sigma} k'(s)^{-\alpha\sigma} \alpha(1 - \alpha)s^{-1} \\
&\quad \cdot \left[\kappa\eta k'(s)^\alpha - (1 - \kappa)(1 - \alpha)k^\alpha k'(s)^{\alpha-1}s \right] d\Psi \\
&= \frac{\alpha\beta}{s} k'(s)^{\alpha(1-\frac{1}{\rho})} \Gamma_2^{\frac{\sigma-\frac{1}{\rho}}{1-\sigma}} \int (\kappa\eta(1 - \alpha) + \alpha)^{-\sigma} [\kappa\eta(1 - \alpha) + \alpha - 1] d\Psi \\
&= \frac{\alpha\beta}{s} k(s)^{\alpha(1-\frac{1}{\rho})} (\tilde{\Gamma}_2 - \tilde{\Gamma}).
\end{aligned}$$

When maximizing steady state utility, $FG(s)$ is equivalent to the derivative of the utility

function with respect to the current period capital stock. Therefore:

$$FG(s) = u_{cy} c_{k(s)}^y k(s)_s + \beta \left(\int c^o(\eta)^{1-\sigma} d\Psi \right)^{\frac{\sigma-\frac{1}{\rho}}{1-\sigma}} \int (c^o(\eta)^{-\sigma}) c_{k'(s)}^o k'(s)_{k(s)} k(s)_s d\Psi,$$

where

$$\begin{aligned} u_{cy} c_{k(s)}^y k(s)_s &= ((1-s)(1-\kappa)(1-\alpha)k(s)^\alpha)^{-\frac{1}{\rho}} (1-s)(1-\kappa)(1-\alpha)\alpha k(s)^{\alpha-1} (1-\kappa)k(s)^\alpha \\ &= \frac{\alpha}{s(1-\alpha)} \left(\frac{1-s}{s} \right)^{1-\frac{1}{\rho}} k'(s)^{1-\frac{1}{\rho}} \end{aligned}$$

and

$$\left(\int c^o(\eta)^{1-\sigma} d\Psi \right)^{\frac{\sigma-\frac{1}{\rho}}{1-\sigma}} = \Gamma_2^{\frac{\sigma-\frac{1}{\rho}}{1-\sigma}} k'(s)^{\alpha(\sigma-\frac{1}{\rho})}$$

and

$$\begin{aligned} &\beta \int c^{0-\sigma} c_{k'(s)}^o k'(s)_{k(s)} k(s)_s d\Psi = \\ &\beta \int (\kappa\eta(1-\alpha) + \alpha)^{-\sigma} k'(s)^{-\alpha\sigma} (\kappa\eta(1-\alpha) + \alpha) d\Psi \alpha k'(s)^{\alpha-1} k'(s) \alpha (1-\kappa) k(s)^{\alpha-1} \\ &= \frac{\alpha^2 \beta}{s(1-\alpha)} k'(s)^{\alpha(1-\sigma)} \Gamma_2. \end{aligned}$$

Therefore:

$$FG(s) = \frac{\alpha}{s(1-\alpha)} \left(\frac{1-s}{s} \right)^{1-\frac{1}{\rho}} k(s)^{1-\frac{1}{\rho}} + \frac{\alpha^2 \beta}{s(1-\alpha)} k(s)^{\alpha(1-\frac{1}{\rho})} \tilde{\Gamma}_2.$$

□

F.6 Decomposition of τ^*

Corollary 9. τ^* can only be decreasing in risk if the effect of $FG(s)$ is sufficiently strong.

Proof. We know that the FOC for s^* follows from

$$PE(s) + CG(s) + FG(s) = 0$$

Now set $FG(s) = 0$. Rewrite from (54)

$$PE(s) + CG(s) = 0 \Leftrightarrow \frac{s}{(1-s)^{\frac{1}{\rho}}} = \alpha\beta\tilde{\zeta}\tilde{\Gamma}_2,$$

which uses $k(s) = (s(1-\kappa)(1-\alpha))^{\frac{1}{1-\alpha}}$ and $\tilde{\zeta} = ((1-\alpha)(1-\kappa))^{\frac{1}{\rho}-1}$. Using the above in (50) gives

$$1 = (1-\tau^*)\frac{\tilde{\Gamma}}{\tilde{\Gamma}_2}$$

and $\frac{\tilde{\Gamma}}{\tilde{\Gamma}_2}$ is unambiguously increasing in risk, see Appendix G.1. Using the above we can thus decompose equation (32) as

$$1 = \underbrace{(1-\tau^*)\frac{\tilde{\Gamma}}{\tilde{\Gamma}_2}}_{\text{from } PE(s)+CG(s)} - \underbrace{(1-\tau^*)\frac{\alpha}{s^*}\frac{\tilde{\Gamma}}{\tilde{\Gamma}_2}}_{\text{from } FG(s)}.$$

□

F.7 Pareto Improving Transitions

Observe that specification (37) nests EZW preferences as a special case. Thus, Proposition 20 and Corollary 7 apply.

G Income Risk and $\Gamma, \Gamma_2, \tilde{\Gamma}, \tilde{\Gamma}_2$

G.1 General Case

In this section we prove Lemma 1 in the main text through two separate Lemmas. For this, recall that the relevant expressions involving idiosyncratic income risk are given by:

$$\begin{aligned}
\Gamma &= \int (\kappa\eta(1-\alpha) + \alpha)^{-\sigma} d\Psi(\eta) \\
\Gamma_2 &= \int (\kappa\eta(1-\alpha) + \alpha)^{1-\sigma} d\Psi(\eta) \\
\tilde{\Gamma} &= \Gamma_2^{\frac{\sigma-\frac{1}{\rho}}{1-\sigma}} \Gamma = v^{\sigma-\frac{1}{\rho}} \Gamma \\
\tilde{\Gamma}_2 &= \Gamma_2^{\frac{\sigma-\frac{1}{\rho}}{1-\sigma}} \Gamma_2 = \Gamma_2^{\frac{1-\frac{1}{\rho}}{1-\sigma}} = v^{1-\frac{1}{\rho}} \\
\frac{\tilde{\Gamma}}{\tilde{\Gamma}_2} &= \frac{\Gamma}{\Gamma_2} \\
v &\equiv \begin{cases} [\int (\alpha + (1-\alpha)\kappa\eta)^{1-\sigma} d\Psi(\eta)]^{\frac{1}{1-\sigma}} & \text{for } \sigma \neq 1 \\ \exp [\int \ln (\alpha + (1-\alpha)\kappa\eta) d\Psi(\eta)] & \text{for } \sigma = 1 \end{cases}
\end{aligned}$$

Furthermore, as in the main text we use the notion of a mean-preserving spread in the random variable η when referring to an increase in risk, that is, formally, random variable η is replaced by $\tilde{\eta} = \eta + \nu$, where ν is a random variable with zero mean and positive variance (and Assumption 1 applies to $\tilde{\eta}$ as well).

Lemma 2. *The certainty equivalent v is decreasing in η -risk.*

Proof. If $\sigma > 1$ ($\sigma < 1$), then $(\alpha + (1-\alpha)\kappa\eta)^{1-\sigma}$ is convex and downward sloping (concave and upward sloping) in η . The certainty equivalent of a convex and downward sloping (respectively, concave and upward sloping) function is decreasing in risk. \square

Lemma 3. *The comparative statics of the other risk terms with respect to a mean-preserving spread in η are given by:*

1. Γ is increasing in η -risk.
2. Γ_2 is increasing (respectively, decreasing) in η -risk if $\sigma > 1$ (respectively $\sigma < 1$).
3. $\tilde{\Gamma}_2$ is increasing (decreasing) in η -risk if $\rho < 1$ ($\rho > 1$).

4. For $\rho < 1$, $\tilde{\Gamma}$ is increasing in η -risk. For $\rho > 1$ we have the following case distinction:

- (a) For $\frac{1}{\sigma} > \rho > 1$, $\tilde{\Gamma}$ unambiguously increases in income risk.
- (b) For $\rho > 1, \rho > \frac{1}{\sigma} > 0$, i.e., $\sigma < \infty$ the effect of η -risk on $\tilde{\Gamma}$ is ambiguous.

Proof. 1. Γ is increasing in η -risk because $(\kappa\eta(1 - \alpha) + \alpha)^{-\sigma}$ is a convex function in η (with the degree of convexity increasing in σ).

2. Γ_2 is increasing (decreasing) in η -risk if $\sigma > 1$ ($\sigma < 1$) because $(\kappa\eta(1 - \alpha) + \alpha)^{1-\sigma}$ is a convex (concave) function of η .

3. $\tilde{\Gamma}_2$ is increasing (decreasing) in η -risk if $\rho < 1$ ($\rho > 1$) because the certainty equivalent v decreases in η -risk and because for $\rho < 1$ ($\rho > 1$) the exponent $1 - \frac{1}{\rho}$ is negative (positive).

4. For $\rho < 1$, $\tilde{\Gamma}$ is increasing in η -risk (sufficient condition). To see this, rewrite $\tilde{\Gamma}$ as

$$\tilde{\Gamma} = \frac{\Gamma}{\Gamma_2^{-\frac{-(1-\sigma)+(1-\frac{1}{\rho})}{1-\sigma}}} = \frac{\Gamma}{\Gamma_2^{1-\frac{1-\frac{1}{\rho}}{1-\sigma}}} = \frac{\Gamma}{\Gamma_2} \Gamma_2^{\frac{1-\frac{1}{\rho}}{1-\sigma}} = \frac{\Gamma}{\Gamma_2} v^{1-\frac{1}{\rho}} \quad (56)$$

Notice that for $\sigma \leq 1$, $\frac{\Gamma}{\Gamma_2}$ is the ratio of the expectation of a strictly convex and a concave function. Hence, for $\sigma \leq 1$ the term $\frac{\Gamma}{\Gamma_2}$ is increasing in risk by Jensen's inequality. For $\sigma > 1$ term $\frac{\Gamma}{\Gamma_2}$ is the ratio of the expectation of two convex functions with the convexity of the function in the numerator, $(\kappa\eta(1 - \alpha) + \alpha)^{-\sigma}$, being stronger than in the denominator, $(\kappa\eta(1 - \alpha) + \alpha)^{1-\sigma}$ as long as $\sigma < \infty$. Therefore, also for $1 < \sigma < \infty$ term $\frac{\Gamma}{\Gamma_2}$ is increasing in risk. For $\sigma = \infty$ term $\frac{\Gamma}{\Gamma_2}$ is equal to 1. Finally, since the certainty equivalent v is decreasing in η -risk, term $v^{1-\frac{1}{\rho}}$ increases in η -risk if and only if $\rho < 1$. For $\rho > 1$ we have the following case distinction:

- (a) For $\frac{1}{\sigma} > \rho > 1$, $\tilde{\Gamma}$ unambiguously increases in η -risk because v decreases in η -risk and $\sigma - \frac{1}{\rho} < 0$.
- (b) For $\rho > 1, \rho > \frac{1}{\sigma} > 0$ the effect of η -risk on $\tilde{\Gamma}$ is ambiguous because v is decreasing in η -risk and $\sigma - \frac{1}{\rho} > 0$ so that $v^{\sigma-\frac{1}{\rho}}$ is decreasing in η -risk whereas Γ is increasing in η -risk. Rewriting $\tilde{\Gamma}$ as in equation (56) does not resolve this ambiguity because term $\frac{\Gamma}{\Gamma_2}$ is increasing in η -risk whereas $v^{1-\frac{1}{\rho}}$ is decreasing in η risk because $1 - \frac{1}{\rho} > 0$.

□

G.2 Expressing Γ -Intervals from Proposition 4 in Terms of Variances

The bounds in Proposition 4 can be given in terms of the variances of the income shock η , to a second-order Taylor approximation of the integral defining Γ . This approximation around $\eta = 1$ gives

$$\Gamma(\alpha, \kappa, \sigma, \Psi) \approx \bar{\Gamma} + \frac{[\kappa(1 - \alpha)]^2}{[\kappa(1 - \alpha) + \alpha]^3} \sigma_\eta^2.$$

With this approximation the interval for intermediate risk, item 2 of Proposition 4, becomes $\sigma_\eta^2 \in (\underline{\sigma}_\eta^2, \overline{\sigma}_\eta^2)$ where

$$\begin{aligned} \underline{\sigma}_\eta^2 &= \frac{(\kappa(1 - \alpha) + \alpha)^3}{(\kappa(1 - \alpha))^2} \left(\frac{1 + \beta}{(1 - \alpha)\beta} - \bar{\Gamma} \right) \\ \overline{\sigma}_\eta^2 &= \frac{(\kappa(1 - \alpha) + \alpha)^3}{(\kappa(1 - \alpha))^2} \left(\frac{1}{((1 - \alpha) - \frac{1}{\bar{\Gamma}})\beta} - \bar{\Gamma} \right) \end{aligned}$$

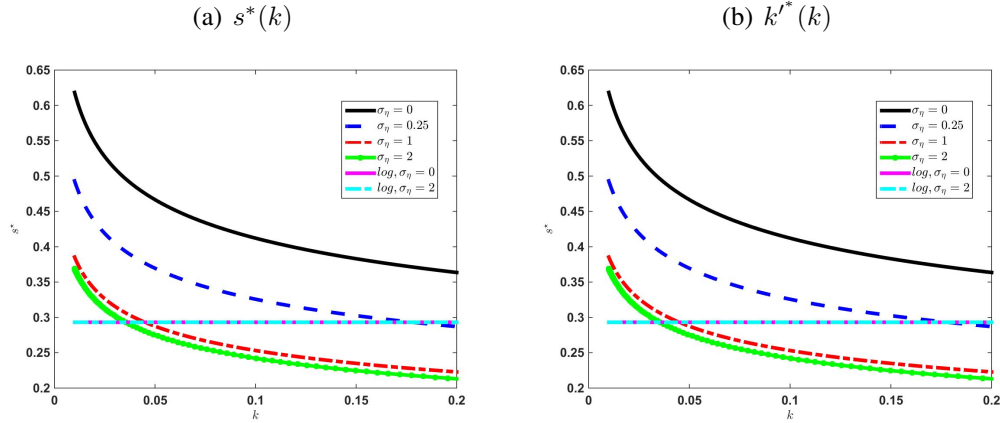
and $\overline{\sigma}_\eta^2 > \underline{\sigma}_\eta^2 > 0$ under the maintained assumption that $\beta < [(1 - \alpha)\bar{\Gamma} - 1]^{-1}$. Thus, all intervals defined in Proposition 4 can be expressed in terms of variances and are non-empty. Also note that if the distribution Ψ is log-normal and thus exclusively determined by its variance (given that the mean is pinned down by the assumption $E(\eta) = 1$), then no second order approximation is necessary in the above argument, but the mapping between the variance bounds and the Γ bounds is algebraically much more involved.

H Further Numerical Results

In Section 6.3 we showed results for an extreme parameterization $\rho = 20, \sigma = 50$ to illustrate hump shaped saving rates in the competitive equilibrium and hump shaped optimal capital income tax rates in the optimal policy. We now first complement this analysis by discussing the associated policy functions. Figure 2 plots the optimal Ramsey saving rate (Panel (a)) and the implied capital stock carried into the next period (Panel (b)) against the capital stock today, for various degrees of income risk (and $\rho = 20, \sigma = 50$). The figure also displays the policy functions for logarithmic utility ($\rho = \sigma = 1$) for $\sigma_{\ln \eta}^2 \in \{0, 2\}$ and confirms that for $\rho = 1$ the optimal saving rate is independent of income risk and of the current capital stock. Relative to this benchmark, and consistent with our steady state findings

in Proposition 9, for an IES $\rho > 1$ the saving policy function is *decreasing* in income risk. Thus, as shown theoretically in the steady state with a high IES the Ramsey government optimally shifts consumption towards the first period of individuals' lives when income risk rises. Panel (a) also shows that the saving rate is a decreasing function of the current capital stock if $\rho > 1$ since an increase in the capital stock raises wages and thus labor income risk when old and lowers returns, thereby leading to a reduction in the saving rate when households are willing to intertemporally substitute consumption. Consequently, the optimal capital stock tomorrow is less elastic to capital today with a high IES, relative to the log-case, as shown in Panel (b).

Figure 2: Policy Functions for $\rho = 20$, $\sigma = 50$ and Log Utility ($\sigma = \rho = 1$)

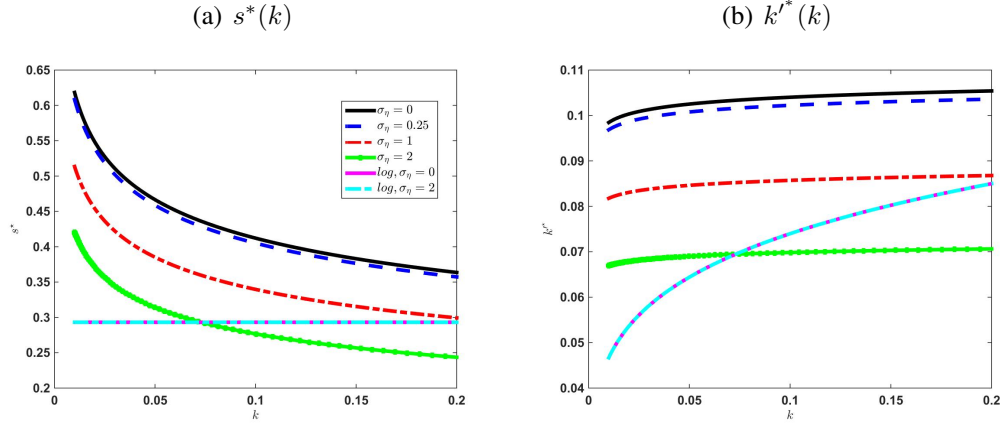


Notes: Optimal saving rate, next period capital stock as function of current k ; for $\rho = 20$, $\sigma = 50$ and for logarithmic utility ($\rho = \sigma = 1$).

Next, we reduce ρ , σ to more conventional values. Results for lower risk aversion $\sigma = 2$ (maintaining $\rho = 20$) are shown in Figures 3–4 and Table 2. Given the lower risk aversion, the optimal policy for the economy with $\sigma_{\ln \eta}^2 = 0.25$ is to implement a capital income subsidy. While the competitive equilibrium saving rate continues to be slightly decreasing in risk, when $\sigma_{\ln \eta}$ increases from 1 to 2, the optimal tax rate is strictly increasing in risk because risk aversion is too low and the future generations effect is not powerful enough to offset the increasing capital income tax.

Results for lower risk aversion $\sigma = 2$ and lower IES of $\rho = 0.5$ are shown in Figures 5–6 and Table 3. As the motive for inter-temporal substitution is now less strong, policy functions for the optimal saving rate are increasing in risk and in the capital stock. With this calibration, the competitive equilibrium saving rate is only too high relative to the long-

Figure 3: Policy Functions for $\rho = 20$, $\sigma = 2$ and Log Utility ($\sigma = \rho = 1$)



Notes: Policy function for optimal saving rate and next period capital stock for $\alpha = 0.2$, $\beta = 0.8$, $\kappa = 0.5$, $\sigma_{\ln \eta} \in \{0, 0.25, 1, 2\}$, $\theta = 0.9$ and $\rho = 20$, $\sigma = 2$ as well as for logarithmic utility ($\rho = \sigma = 1$).

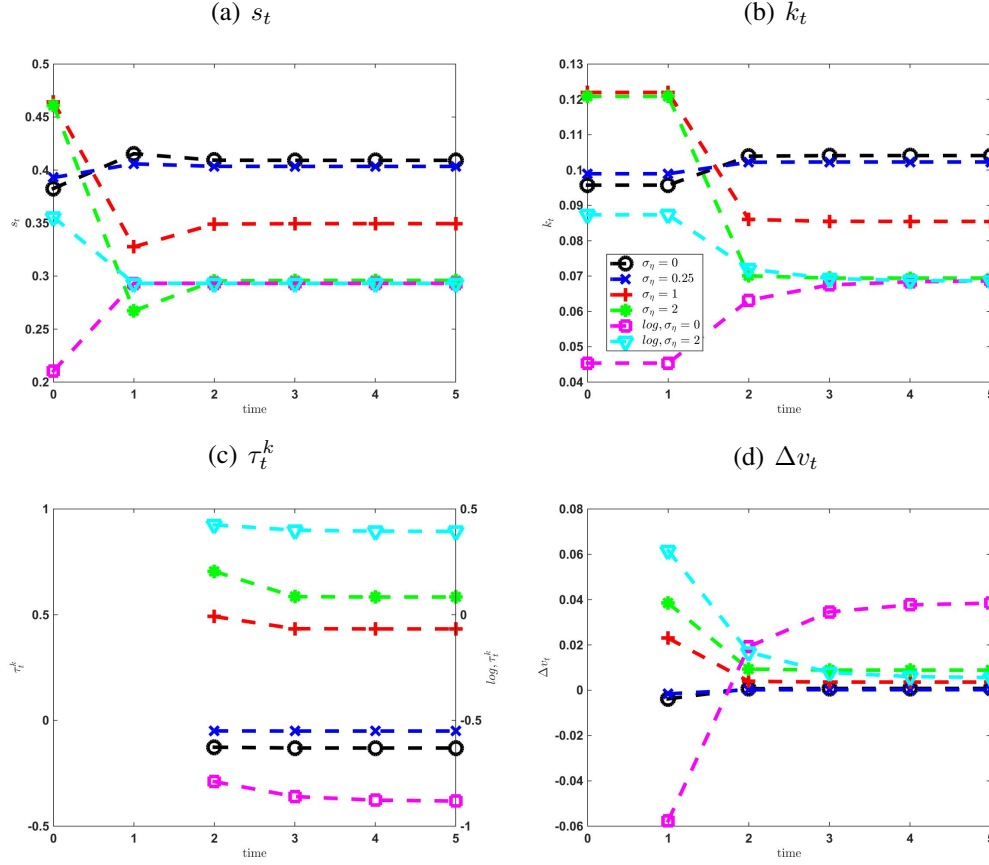
Table 2: CE and Optimal Long-Run Saving & Capital Income Tax Rates: Low RA

	s^{CE}	s_{∞}^*	$\tau_{\infty}^{k^*}$
EZW-Preferences with $\rho = 20$, $\sigma = 2$			
$\sigma_{\ln \eta} = 0$	0.38	0.41	-0.13
$\sigma_{\ln \eta} = 0.25$	0.39	0.4	-0.05
$\sigma_{\ln \eta} = 1$	0.464	0.35	0.43
$\sigma_{\ln \eta} = 2$	0.460	0.3	0.58

Notes: Saving rates in the initial competitive equilibrium, s^{CE} , and optimal long-run saving, s_{∞}^* , and capital income tax rates $\tau_{\infty}^{k^*}$ for $\alpha = 0.2$, $\beta = 0.8$, $\kappa = 0.5$, $\sigma_{\ln \eta} \in \{0, 0.25, 1, 2\}$, $\theta = 0.9$ and $\rho = 20$, $\sigma = 2$.

run optimum for high income risk ($\sigma_{\ln \eta}^2 = 2$), and only this economy experiences capital income taxation along the transition. It also features a Pareto improvement from the tax reform.

Figure 4: Policy Transition for $\rho = 20$, $\sigma = 2$ and Log Utility ($\sigma = \rho = 1$)



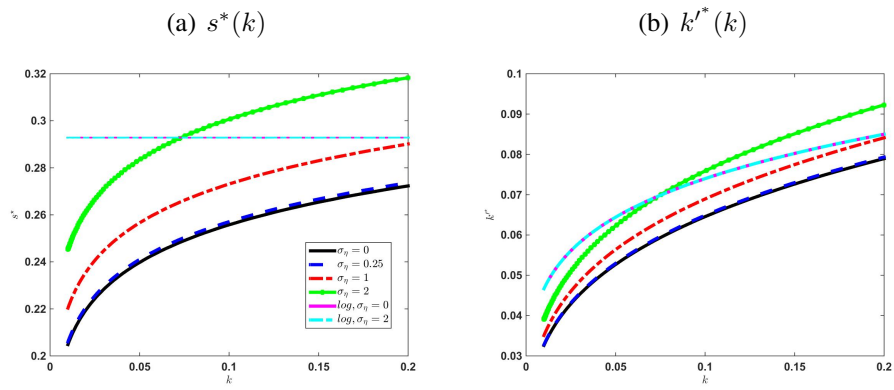
Notes: Initial and optimal saving rate, capital stock, optimal capital income tax rate and changes in lifetime utility in transition for $\alpha = 0.2$, $\beta = 0.8$, $\kappa = 0.5$, $\sigma_{\ln \eta} \in \{0, 0.25, 1, 2\}$, $\theta = 0.9$ and $\rho = 20$, $\sigma = 50$ as well as for logarithmic utility ($\rho = \sigma = 1$).

Table 3: CE and Long Run Optimal Saving & Capital Income Tax Rate: Low IES, RA

	s^{CE}	s_T^*	τ_T^*
EZW-Preferences with $\rho = 0.5$, $\sigma = 2$			
$\sigma_{\ln \eta} = 0$	0.13	0.24	-2.02
$\sigma_{\ln \eta} = 0.25$	0.14	0.24	-1.78
$\sigma_{\ln \eta} = 1$	0.24	0.26	-0.26
$\sigma_{\ln \eta} = 2$	0.36	0.29	0.55

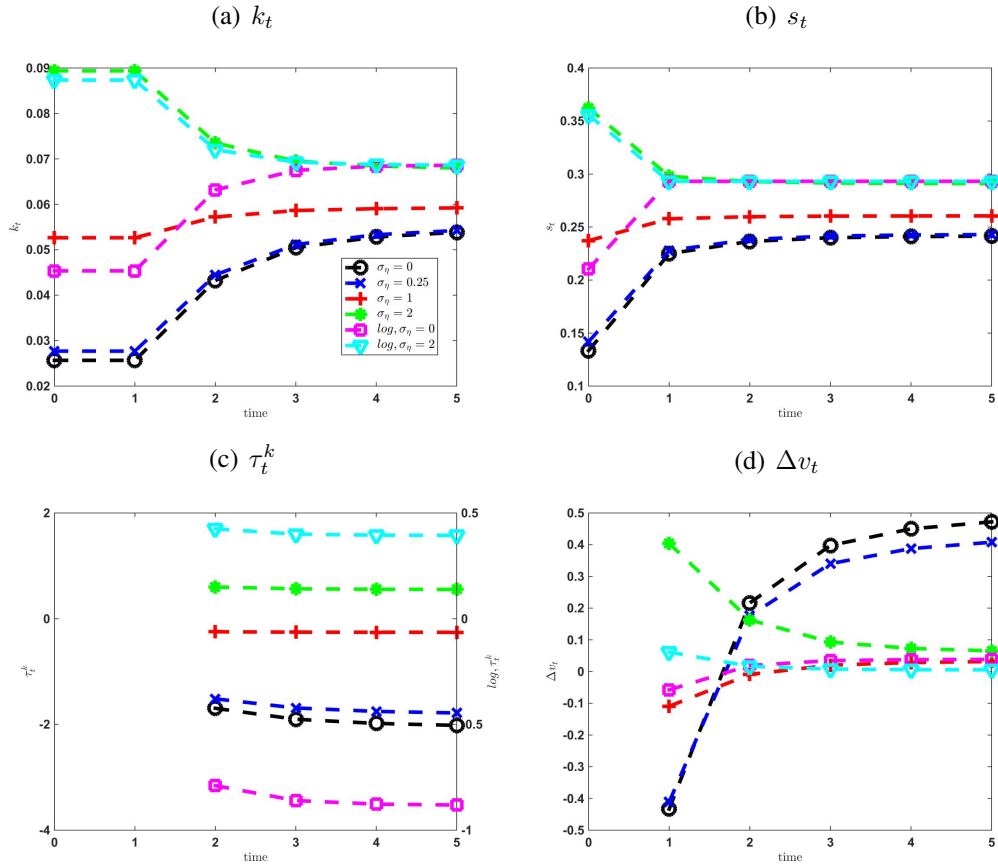
Notes: Saving rates in the initial competitive equilibrium, s^{CE} , and optimal long-run saving, s_∞^* , and capital income tax rates τ_∞^{k*} for $\alpha = 0.2$, $\beta = 0.8$, $\kappa = 0.5$, $\sigma_{\ln \eta} \in \{0, 0.25, 1, 2\}$, $\theta = 0.9$ and $\rho = 0.5$, $\sigma = 2$.

Figure 5: Policy Functions for $\rho = 0.5$, $\sigma = 2$ and Log Utility ($\sigma = \rho = 1$)



Notes: Policy function for optimal saving rate and next period capital stock for $\alpha = 0.2$, $\beta = 0.8$, $\kappa = 0.5$, $\sigma_{\ln \eta} \in \{0, 0.25, 1, 2\}$, $\theta = 0.9$ and $\rho = 0.5$, $\sigma = 2$ as well as for logarithmic utility ($\rho = \sigma = 1$).

Figure 6: Policy Transition for EZW with $\rho = 0.5$, $\sigma = 2$ and Log Utility



Notes: Capital stock, saving rate, capital income tax rate and changes in lifetime utility in transition for $\alpha = 0.2$, $\beta = 0.8$, $\kappa = 0.5$, $\sigma_{\ln \eta} \in \{0, 0.25, 1, 4\}$, $\theta = 0.9$ and $\sigma = 2$, $\rho = 0.5$ as well as for logarithmic utility.